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Abstract

By introducing the busy beaver competition of Turing machines, in 1962, Rado defined noncomputable functions on positive integers. The study of these functions and variants leads to many mathematical challenges. This article takes up the following one: How can a small Turing machine manage to produce very big numbers? It provides the following answer: mostly by simulating Collatz-like functions, that are generalizations of the famous $3x+1$ function. These functions, like the $3x+1$ function, lead to new unsolved problems in number theory.

Keywords: Turing machine, busy beaver, Collatz-like functions.

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1 Introduction

1.1 A well defined noncomputable function

It is easy to define a noncomputable function on nonnegative integers. Indeed, given a programming language, you produce a systematic list of the programs for functions, and, by diagonalization, you define a function whose output, on input n, is different from the output of the nth program. This simple definition raises many problems: Which programming language? How to list the programs? How to choose the output?

In 1962, Rado [38] gave a practical solution by defining the *busy beaver game*, also called now the *busy beaver competition*. Consider all Turing machines on one infinite tape, with n

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states (plus a special halting state), and two symbols (1, and the blank symbol 0), and launch all of them on a blank tape. Define $S(n)$ as the maximum number of computation steps made by such a machine before it stops, and define $\Sigma(n)$ as the maximum number of symbols 1 left on the tape by a machine when it stops. Then functions S and Σ are noncomputable, and, moreover, grow faster than any computable function, that is, for any computable function f, there exists an integer N such that, for any $n \geq N$, $S(n) > \Sigma(n) > f(n)$.

More than fifty years later, no better choice has been found for a practical noncomputable function. Only variants of Rado's definition have been proposed. So, in 1988, Brady [5] defined similar functions $S(n, m)$ and $\Sigma(n, m)$ for $n \times m$ Turing machines, that is Turing machines with n states and m symbols. He also introduced analogous functions for two-dimensional Turing machines and "turNing machines", later resumed and expanded by Tim Hutton [13]. Bátfai [1, 2] relaxed the rule about head moving, by allowing the head to stand still.

In this article, we will consider functions $S(n, m)$ and $\Sigma(n, m)$. Recall that $S(n) = S(n, 2)$ and $\Sigma(n) = \Sigma(n, 2)$.

1.2 Computing the values of noncomputable functions

The busy beaver functions S and Σ are explicitely defined, and it is possible to compute $S(n, m)$ and $\Sigma(n, m)$ for small n and m. In the first article on busy beavers, Rado [38] gave $\Sigma(2) = 2$ and $\Sigma(3) \geq 6$. These results show that two problems are at stake:

- **Problem 1**: To give lower bounds on $S(n, m)$ and $\Sigma(n, m)$ by finding Turing machines with high scores.
- **Problem 2**: To compute $S(n, m)$ and $\Sigma(n, m)$ by proving that no Turing machines do better than the best known ones.

Problem 1 can be tackled either by hand search, as did, for example, Green [11] and Lynn [26], or by computer search, using acceleration techniques of computation and, for example, simulated annealing, as did T. and S. Ligocki [21].

Solving Problem 2 requires more work to be done: clever enumeration of $n \times m$ Turing machines, simulation of computation with acceleration techniques, proofs of non-halting for the machines that do not halt.

Currently, the following results are known (see Michel [32, 33] for a historical survey):

- $S(2) = 6$ and $\Sigma(2) = 4$ (Rado [38]),
- $S(3) = 21$ and $\Sigma(3) = 6$ (Lin and Rado [25]),
- $S(4) = 107$ and $\Sigma(4) = 13$ (Brady [3, 4], Machlin and Stout [27]),
- $S(5) \geq 47,176,870$ and $\Sigma(5) \geq 4098$ (Marxen and Buntrock [29]),
- $S(6) > 7.4 \times 10^{36534}$ and $\Sigma(6) > 3.5 \times 10^{18267}$ (P. Kropitz in 2010),
- $S(2, 3) = 38$ and $\Sigma(2, 3) = 9$ (Lafitte and Papazian [16]),
- $S(3, 3) > 1.1 \times 10^{17}$ and $\Sigma(3, 3) \geq 347,676,383$ (T. and S. Ligocki in 2007),

- $S(4,3) > 1.0 \times 10^{14072}$ and $\Sigma(4,3) > 1.3 \times 10^{7036}$ (T. and S. Ligocki in 2008),
- $S(2,4) \geq 3,932,964$ and $\Sigma(2,4) \geq 2050$ (T. and S. Ligocki in 2005),
- $S(3,4) > 5.2 \times 10^{13036}$ and $\Sigma(3,4) > 3.7 \times 10^{6518}$ (T. and S. Ligocki in 2007),
- $S(2,5) > 1.9 \times 10^{704}$ and $\Sigma(2,5) > 1.7 \times 10^{352}$ (T. and S. Ligocki in 2007),
- $S(2,6) > 2.4 \times 10^{9866}$ and $\Sigma(2,6) > 1.9 \times 10^{4933}$ (T. and S. Ligocki in 2008).

In order to achieve these results, many computational and mathematical challenges had to be taken up.

A. Computational challenges.

- **A1**. To generate all $n \times m$ Turing machines, or rather, to treat all cases without having to generate all $n \times m$ Turing machines.
- **A2**. To simulate the computation of a machine by using acceleration techniques (see Marxen and Buntrock [29], Marxen [28]).
- **A3**. To gave automatic proofs that non-halting machines do not halt (see Brady [4], Marxen and Buntrock [29], Machlin and Stout [27], Hertel [12], Lafitte and Papazian [16]).

B. Mathematical challenges.

- **B1**. To prove by hand that a non-halting machine that resists the computational proof does not halt.
- **B2**. To understand how the Turing machines that reach high scores manage to do it.

1.3 Facing open problems in number theory

Let us come back to mathematical challenge *B1*. For example, the computational study of 5×2 Turing machines by Marxen and Buntrock [29], Skelet [10] and Hertel [12] left holdouts that needed to be analyzed by hand. Marxen and Buntrock [29] gave an unsettled 5×2 Turing machine, named #4, that turned out to never halt, by an intricate analysis.

Actually, the halting problem for Turing machines launched on a blank tape is m-complete, and this implies that this problem is as hard as the problem of the provability of a mathematical statement in a logical theory such as ZFC (Zermelo Fraenkel set theory with axiom of choice). So, when Turing machines with more and more states and symbols are studied, potentially all theorems of mathematics will be met. When more and more non-halting Turing machines are studied to be proved non-halting, one has to expect to face hard open problems in mathematics, that is problems that current mathematical knowledge can't settle.

Consider now mathematical challenge *B2*, which is the very subject of this article.

From 1983 to 1989, several 5×2 Turing machines with high scores were discovered by Uwe Schult, by George Uhing, and by Heiner Marxen and Jürgen Buntrock. Michel [30, 31] analyzed some of these machines and found that their behavior is *Collatz-like*, which implies that the halting problems on general inputs for these machines are open problems in number theory (see Table 1).

From 2005 and 2007, many 3×3, 2×4 and 2×5 Turing machines with high scores were discovered, mainly by two teams: the French one of Grégory Lafitte and Christophe Papazian, and the father-and-son collaboration of Terry and Shawn Ligocki. Collatz-like behavior of these champions seems to be the rule (see Tables 3, 4 and 5).

However, the behaviors of 6×2 Turing machines display some variety. Many machines were discovered, from 1990 to 2010, by Heiner Marxen and Jürgen Buntrock, by Terry and Shawn Ligocki and by Pavel Kropitz. The analyses of some of these machines, by Robert Munafo, Clive Tooth, Shawn Ligocki and the author, show that the behaviors can be Collatz-like, exponential Collatz-like, loosely Collatz-like, or definitely not Collatz-like. Almost all of them raise open problems (see Table 2).

Note: The Turing machines listed in Tables 1–5 are those for which an analysis is known by the author. The machines without references for the study of behavior were analyzed by the author [32, 34]. Many other machines are waiting for their analyses.

1.4 Collatz functions, Collatz-like functions and other functions

The $3x+1$ *function*, or *Collatz function*, is the function T on positive integers defined by

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (3n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

This function is famous because, when it is iterated on a positive integer, it seems to lead to the loop $2, 1, 2, 1, \ldots$. Is it always true? This is an open problem. See Lagarias [17, 18, 19, 20] for more information.

It is natural to generalize the definition of the $3x+1$ function by replacing n even, n odd by $n \equiv 0, \ldots, d-1 \pmod d$, and by replacing $n/2$, $(3n+1)/2$ by $an + b$ for rational numbers a, b. Unfortunately, no name for such functions is currently taken for granted. Formal definitions were given by Rawsthorne [39], who proposed *Collatz-type iteration functions*, by Buttsworth and Matthews [6], who proposed *generalized Collatz mappings*, by Kaščák [14], who proposed *one-state linear operator algorithms (OLOA)*, and by Kohl [15], who proposed *residue-class-wise affine functions (RCWA)*. Without giving a formal definition, Lagarias [17] proposed *periodically linear functions*, and Wagon [41] proposed *Collatz-like functions*.

We will choose the following definitions.

Definition 1.1 *A mapping $f : \mathbb{Z} \to \mathbb{Z}$ is a* generalized Collatz mapping *if there exists an integer $d \geq 2$ such that the following three equivalent conditions are satisfied:*

(i) (see [42, p.14]) There exist rational numbers $q_0, \ldots, q_{d-1}$, $r_0, \ldots, r_{d-1}$, such that, for all i, $0 \leq i \leq d-1$, we have $q_i d \in \mathbb{Z}$, $q_i i + r_i \in \mathbb{Z}$, and, for all $n \in \mathbb{Z}$, $f(n) = q_i n + r_i$ if $n \equiv i \pmod d$.

(ii) (see [6]) There exist integers $m_0, \ldots, m_{d-1}$, $p_0, \ldots, p_{d-1}$, such that, for all i, $0 \leq i \leq d-1$, we have $p_i \equiv i m_i \pmod d$ and, for all $n \in \mathbb{Z}$, $f(n) = (m_i n - p_i)/d$ if $n \equiv i \pmod d$.

(iii) There exist integers $a_0, \ldots, a_{d-1}$, $b_0, \ldots, b_{d-1}$, such that we have, for all i, $0 \leq i \leq d-1$, for all $n \in \mathbb{Z}$, $f(dn + i) = a_i n + b_i$.

These definitions are easily seen to be equivalent: we have $a_i = m_i = q_i d$ and $b_i = (im_i - p_i)/d = q_i i + r_i$.

The definitions above concern total functions, but, in this article, we always deal with partial functions and functions with parameters, so we introduce the following definitions.

Definition 1.2 *A partial function $f : \mathbb{Z} \to \mathbb{Z}$ is a* generalized Collatz function, *or a* pure Collatz-like function (without parameter) *if, in the previous definition, $f(dn + i)$ can be undefined for one or many i, $0 \leq i \leq d - 1$.*

Definition 1.3 *A partial function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is a* pure Collatz-like function with parameter *if there exist an integer $d \geq 2$, integers $a_0, \ldots, a_{d-1}$, $b_0, \ldots, b_{d-1}$, a set S of integers and a function $p : \{0, \ldots, d-1\} \times S \to S$ such that, for all i, $0 \leq i \leq d - 1$, for all $n \in \mathbb{Z}$, for all $s \in S$, $f(dn + i, s) = (a_i n + b_i, p(i, s))$ or is undefined.*

Definition 1.4 *If, in the definitions above, $a_i = a$ for all i, $0 \leq i \leq d - 1$, we say that f is pure Collatz-like* of type $d \to a$.

We also need to define a new type of function, as follows.

Definition 1.5 *A partial function $f : \mathbb{Z} \to \mathbb{Z}$ is an* exponential Collatz-like function *if there exist integers $d, p \geq 2$, integers $a_0, \ldots, a_{d-1}$, $b_0, \ldots, b_{d-1}$, $c_0, \ldots, c_{d-1}$, such that, for all i, $0 \leq i \leq d-1$, all $n \in \mathbb{Z}$, $f(dn+i) = (a_i p^n + b_i)/c_i$ or is undefined. In this definition, integers p, a_i, b_i, c_i are chosen such that $(a_i p^n + b_i)/c_i$ is an integer for all $n \in \mathbb{Z}$.*

Currently, no study of this type of function is known. Note that iterates $f(n)$, $f^2(n), \ldots$ grow much faster than for pure Collatz-like functions.

1.5 From Collatz-like functions to high scores

The Turing machines studied in this article have behaviors modeled on iterations of functions, where halting configurations correspond to undefined values of functions.

In Section 3, we present a 3×3 Turing machine M_1 whose behavior is pure Collatz-like, of type $8 \to 14$. In Section 4, we present a 2×4 Turing machine M_2 whose behavior is pure Collatz-like with parameter, of type $3 \to 5$. In Section 5, we present a 2×5 Turing machine M_3 whose behavior is pure Collatz-like with parameter, of type $2 \to 3$. Thus, the halting problem for machines M_1, M_2 and M_3 depends on an open problem about iterating Collatz-like functions.

In Section 6, we present a 6×2 Turing machine M_4 whose behavior is exponential Collatz-like.

In Section 7, we present a 6×2 Turing machine M_5 whose behavior depends on iterating a partial function $g_5(n, p)$. Without being Collatz-like, this function seems to share some properties with Collatz-like functions.

In Section 8, we present a 6×2 Turing machine M_6 whose behavior looks like a loosely Collatz-like behavior with parameter, of type $2 \to 5$. The novelty is that a potentially infinite set of rules seems to be necessary to describe the behavior of the machine on inputs $00x$, $x \in \{0, 1\}^*$. A string $x \in \{0, 1\}^*$ ending with symbol 1 can be taken as the binary writing of a number p, read in the opposite direction, so $x = \mathrm{R}(\mathrm{bin}(p))$, where $\mathrm{bin}(p)$ is the usual binary writing of p, and $\mathrm{R}(w)$ is the reverse of string w, that is $\mathrm{R}(w_1 \dots w_n) = w_n \dots w_1$. In Table 2 we write "R(bin(p))" to indicate the machines with a behavior involving an infinite set of rules. Of course, only a finite subset of these rules are used when the machine is launched on a blank tape.

In Section 9, we present a 6×2 Turing machine M_7 whose behavior on the blank tape depends on configurations $C(n)$ all of them provably leading to a halting configuration. We present such a machine to show how a Turing machine can take a long time to stop without calling for Collatz-like functions.

2 Preliminaries

A *Turing machine* involved in the busy beaver competition is defined as follows. It has a tape made of cells, infinite in both directions. Each cell contains a *symbol*, and a head can read and write a symbol on a cell. The Turing machine can be in a finite number of *states*. A computation of the Turing machine is a sequence of steps. In a step of computation, according to the current state and the symbol read by the head on the current cell, the head writes a symbol on the cell, moves to the next cell on the right side or on the left side, and the machine enters a new state.

Formally, a Turing machine $M = (Q, \Gamma, \delta)$ has a finite set of states $Q = \{A, B, \dots\}$, a finite set of symbols $\Gamma = \{0, 1, \dots\}$, and a *transition function* (or *next move function*) δ, which is a mapping

$$\delta : Q \times \Gamma \to (\Gamma \times \{L, R\} \times Q) \cup \{(1, R, H)\}.$$

If $\delta(q, a) = (b, D, q')$, then the Turing machine, when it is in state q reading symbol a on the current cell, writes symbol b instead of a on this cell, moves one cell left if $D = L$, one cell right if $D = R$, and comes in state q'. The transition function is usually given by a *transition table*.

There a special state A, called the *initial state*, and a special symbol 0, called the *blank symbol*. In the busy beaver competition, at the beginning of a computation, the Turing machine is in state A, and the tape is blank, that is all the cells of the tape contain the blank symbol. There is another state H, the *halting state*, not in the set Q of states. When a Turing machine comes in this state, the computation stops. We impose that, at the last step, the machine writes 1, moves right, and comes in state H.

A *word* is a finite string of symbols. The set of words with symbols in the set Γ is denoted by Γ^*. The number of symbols in a word $x \in \Gamma^*$ is called the *length* of x and is denoted by $|x|$. The *empty word* is the word of length zero, denoted by λ. If $x \in \Gamma^*$, and $n \geq 0$, x^n is the word $xx \dots x$, where x is repeated n times, that is, formally: $x^0 = \lambda$, $x^1 = x$ and $x^{n+1} = x^n x$. An infinite to the left string of 0 is denoted by ${}^\omega 0$, and an infinite to the right string of 0 is denoted by 0^ω.

A *configuration* is a way to encode the symbols on the tape, the state, and the cell currently read by the head. The Turing machine is in configuration ${}^\omega 0 x (Sa) y 0^\omega$, with $S \in Q \cup \{H\}$, $a \in \Gamma$, $x, y \in \Gamma^*$, if the word xay is written on the tape, the state is S, and the head is reading symbol a. Since, at the beginning of the computation, the state is A and the tape is blank, the initial configuration is ${}^\omega 0(A0)0^\omega$. If the state is H, the configuration is halting. We also consider configurations $x(Sa)y$ with finite length. If the computation from

Machine	Behavior	Study of behavior
January 1983 Uwe Schult $\sigma = 501$ $s = 134{,}467$	Pure Collatz-like ($4 \to 9$) without parameter 7 rules 7 transitions	Robinson, cited in [9]. Michel [30]
December 1984 George Uhing $\sigma = 1915$ $s = 2{,}133{,}492$	Pure Collatz-like ($3 \to 8$) with parameter 5 rules 9 transitions	Michel [30]
February 1986 George Uhing $\sigma = 1471$ $s = 2{,}358{,}064$	Pure Collatz-like ($8 \to 15$) without parameter 5 rules 11 transitions	
August 1989 Marxen, Buntrock $\sigma = 4098$ $s = 11{,}798{,}826$	Pure Collatz-like ($3 \to 5$) without parameter 3 rules 15 transitions	Michel [30]
September 1989 Marxen, Buntrock $\sigma = 4097$ $s = 23{,}554{,}764$	Pure Collatz-like ($3 \to 5$) without parameter 3 rules 15 transitions	Michel [30]
September 1989 Marxen, Buntrock $\sigma = 4098$ $s = 47{,}176{,}870$	Pure Collatz-like ($3 \to 5$) without parameter 3 rules 15 transitions	Michel [31]

Table 1: Study of behavior of 5×2 machines

Machine	Behavior	Study of behavior
September 1997 Marxen and Buntrock $s > 8.6 \times 10^{15}$	Pure Collatz-like $(4 \to 10)$ without parameter 5 rules 21 transitions	Munafo [36]
October 2000 Marxen and Buntrock (machine o) $s > 6.1 \times 10^{119}$	R(bin(p)) $(2 \to 3)$ 9 rules 337 transitions	
October 2000 Marxen and Buntrock (machine q) $s > 6.1 \times 10^{925}$	All $C(n)$ stop 4 rules 5 transitions	Munafo [37] Section 9
March 2001 Marxen and Buntrock $s > 3.0 \times 10^{1730}$	R(bin(p)) $(2 \to 3)$ 20 rules 4911 transitions	Tooth [40]
November 2007 T. and S. Ligocki $s > 8.9 \times 10^{1762}$	R(bin(p)) $(2 \to 5)$ 12 rules 3346 transitions	Section 8
December 2007 T. and S. Ligocki $s > 2.5 \times 10^{2879}$	R(bin(p)) $(4 \to 6)$ 18 rules 11026 transitions	
May 2010 Pavel Kropitz $s > 3.8 \times 10^{21132}$	Unclassifiable 6 rules 22158 transitions	S. Ligocki [24] Section 7
June 2010 Pavel Kropitz $s > 7.4 \times 10^{36534}$	Exponential Collatz-like without parameter 4 rules 5 transitions	Section 6

Table 2: Study of behavior of 6×2 machines

Machine	Behavior	Study of behavior
December 2004 Brady $s = 92{,}649{,}163$	Pure Collatz-like ($2 \to 5$) with parameter 5 rules 11 transitions	
July 2005 Souris $\sigma = 36089$ $s = 310{,}341{,}163$	Pure Collatz-like ($2 \to 5$) with parameter 5 rules 12 transitions	
July 2005 Souris $s = 544{,}884{,}219$	Pure Collatz-like ($3 \to 7$) with parameter 7 rules 12 transitions	
August 2005 Lafitte and Papazian $s > 4.9 \times 10^9$	Pure collatz-like ($4 \to 7$) with parameter 8 rules 21 transitions	
September 2005 Lafitte and Papazian $s > 9.8 \times 10^{11}$	Pure Collatz-like ($4 \to 7$) with parameter 7 rules 24 transitions	
April 2006 Lafitte and Papazian $s > 4.1 \times 10^{12}$	Pure Collatz-like ($2 \to 5$) with parameter 5 rules 16 transitions	
August 2006 T. and S. Ligocki $s > 4.3 \times 10^{15}$	Pure Collatz-like ($2 \to 5$) with parameter 4 rules 20 transitions	S. Ligocki [23]
November 2007 T. and S. Ligocki $s > 1.1 \times 10^{17}$	Pure Collatz-like ($8 \to 14$) without parameter 9 rules 34 transitions	Section 3

Table 3: Study of behavior of 3×3 machines

Machine	Behavior	Study of behavior
1988 Brady $s = 7195$	Pure Collatz-like ($3 \to 5$) with parameter 6 rules 7 transitions	
February 2005 T. and S. Ligocki $s = 3{,}932{,}964$	Pure Collatz-like ($3 \to 5$) with parameter 7 rules 14 transitions	Section 4

Table 4: Study of behavior of 2×4 machines

Machine	Behavior	Study of behavior
October 2005 Lafitte, Papazian $s > 9.1 \times 10^{11}$	Pure Collatz-like ($2 \to 5$) with parameter 7 rules 15 transitions	
December 2005 Lafitte, Papazian $s > 9.2 \times 10^{11}$	Pure Collatz-like ($2 \to 5$) with parameter 5 rules 14 transitions	
May 2006 Lafitte, Papazian $s > 3.7 \times 10^{12}$	Pure Collatz-like ($3 \to 4$) with parameter 7 rules 45 transitions	
June 2006 Lafitte, Papazian $s > 1.4 \times 10^{13}$	Pure Collatz-like ($2 \to 3$) with parameter 9 rules 36 transitions	
August 2006 T. and S. Ligocki $s > 7.0 \times 10^{21}$	Pure Collatz-like ($2 \to 5$) with parameter 9 rules 30 transitions	S. Ligocki [22]
November 2007 T. and S. Ligocki $s > 1.9 \times 10^{704}$	Pure Collatz-like ($2 \to 3$) with parameter 17 rules 2002 transitions	Section 5

Table 5: Study of behavior of 2×5 machines

M_1	0	1	2
A	1RB	2LA	1LC
B	0LA	2RB	1LB
C	1RH	1RA	1RC

Table 6: Machine M_1 discovered in November 2007 by T. and S. Ligocki

configuration C_1 to configuration C_2 takes t steps, we write $C_1 \vdash (t)\ C_2$, and t is said to be the *time* taken by the machine to go from C_1 to C_2. If C_2 is a halting configuration, we also write $C_1 \vdash (t)$ END. We write $C_1 \vdash (\ )\ C_2$ if the time is not specified. If C_1 and C_2 are configurations with finite length, then they refer to the same part of the tape. For example, $(A0)0 \vdash (1)\ 1(B0)$ if $\delta(A, 0) = (1, R, B)$.

A Turing machine M computes a partial function $f_M : \Gamma^* \to \Gamma^*$ as follows. let $x = x_1 \dots x_n \in \Gamma^*$. Then x becomes an input for M by considering the computation of M on initial configuration ${}^\omega 0(Ax_1)x_2 \dots x_n 0^\omega$. If M never stops on this configuration, then $f_M(x)$ is undefined. If M stops, in configuration ${}^\omega 0y(Ha)z0^\omega$, with $a \in \Gamma$, $y, z \in \Gamma^*$, then the output $f_M(x)$ is defined from this configuration by a suitable convention. The *halting set* is $\{x \in \Gamma^* : f_M(x) \text{ is defined}\}$. The *halting problem* for machine M is the problem consisting in determining the halting set. Note that the Turing machines with two symbols 0 and 1 are powerful enough to compute any computable function, and their halting sets can be any computably enumerable (also called recursively enumerable) set.

A Turing machine with n states and m symbols is called a $n \times m$ machine. The set of $n \times m$ machines is denoted by TM(n,m). With our definition of the transition function, there are $(2nm + 1)^{nm}$ machines in the set TM(n,m). In the *busy beaver competition*, for fixed numbers of states n and symbols m, all the $(2nm + 1)^{nm}$ Turing machines in TM(n,m) are launched on the blank tape. Some of them never stop. Those which stop are called *busy beaver*. Each busy beaver takes some time to stop, and leaves some non-blank symbols on the tape, so busy beavers are involved in two competitions: to take the longest time before stopping, and to leave the greatest number of non-blank symbols on the tape when stopping. The time taken by Turing machine M to stop is denoted by $s(M)$, and the number of non-blank symbols left by M when it stops is denoted by $\sigma(M)$. The *busy beaver functions* are defined by

$$S(n, m) = \max\{s(M) : M \text{ is a busy beaver with } n \text{ states and } m \text{ symbols}\}$$

$$\Sigma(n, m) = \max\{\sigma(M) : M \text{ is a busy beaver with } n \text{ states and } m \text{ symbols}\}$$

Rado [38] initially defined functions $S(n) = S(n, 2)$ and $\Sigma(n) = \Sigma(n, 2)$ for Turing machines with n states and two symbols.

3 Pure Collatz-like behavior

Let M_1 be the 3×3 Turing machine defined by Table 6

We have $s(M_1) = 119{,}112{,}334{,}170{,}342{,}540$ and $\sigma(M_1) = 374{,}676{,}383$.

This machine is the current champion for the busy beaver competition for 3×3 machines. It was discovered in November 2007 by Terry and Shawn Ligocki, who wrote (email on

November, 9th) that they enumerated all the 3×3 machines and applied the techniques of acceleration and proof systems originally developed by Marxen and Buntrock.

The following theorem gives the rules that enable Turing machine M_1 to reach a halting configuration from a blank tape.

Theorem 3.1 *Let* $C(n) = {}^{\omega}0(A0)2^n0^{\omega}$. *Then*

(a) ${}^{\omega}0(A0)0^{\omega} \vdash (3) \quad C(1)$,

and, for all $k \geq 0$,

(b) $C(8k+1) \vdash (112k^2 + 116k + 13) \quad C(14k+3)$,
(c) $C(8k+2) \vdash (112k^2 + 144k + 38) \quad C(14k+7)$,
(d) $C(8k+3) \vdash (112k^2 + 172k + 54) \quad C(14k+8)$,
(e) $C(8k+4) \vdash (112k^2 + 200k + 74) \quad C(14k+9)$,
(f) $C(8k+5) \vdash (112k^2 + 228k + 97) \quad {}^{\omega}01(H1)2^{14k+9}0^{\omega}$,
(g) $C(8k+6) \vdash (112k^2 + 256k + 139) \quad C(14k+14)$,
(h) $C(8k+7) \vdash (112k^2 + 284k + 169) \quad C(14k+15)$,
(i) $C(8k+8) \vdash (112k^2 + 312k + 203) \quad C(14k+16)$.

Proof. A direct inspection of the transition table gives

(1) $0(A0)0 \vdash (3) \quad (A0)20$,
(2) $0^3(A0)2^5 \vdash (53) \quad (B1)1^8$,
(3) $0(A1) \vdash (1) \quad (A0)2$,
(4) $1(A1) \vdash (1) \quad (A1)2$,
(5) $02(A1) \vdash (3) \quad 1(H1)2$,
(6) $12(A1) \vdash (4) \quad (A1)22$,
(7) $22(A1) \vdash (8) \quad (A1)22$,
(8) $2(B1)0^2 \vdash (7) \quad 11(A1)0$,
(9) $(B1)1 \vdash (1) \quad 2(B1)$,
(10) $(B1)2 \vdash (1) \quad 2(B2)$,
(11) $0^3(B2) \vdash (14) \quad 1^3(B1)$,
(12) $1(B2) \vdash (1) \quad (B1)1$,
(13) $2(B2) \vdash (1) \quad (B2)1$.

From this point, k will be an integer, $k \geq 0$.

Iterating, respectively, (4), (7), (9) and (13) gives

(14) $1^k(A1) \vdash (k) \quad (A1)2^k$,
(15) $2^{2k}(A1) \vdash (8k) \quad (A1)2^{2k}$,
(16) $(B1)1^k \vdash (k) \quad 2^k(B1)$,
(17) $2^k(B2) \vdash (k) \quad (B2)1^k$.

Using consecutively (16), (10), (17) and (12), we get

(18) $1(B1)1^k2 \vdash (2k+3) \quad (B1)1^{k+2}$.

Using (16), (10), (17) and (11), we get

(19) $0^3(B1)1^k2 \vdash (2k+16) \quad 1^3(B1)1^{k+1}$.

Using (19) and three times (18), we get

(20) $0^3(B1)1^k2^4 \vdash (8k+43) \quad (B1)1^{k+7}$.

For any $n \geq 0$, by induction on k, using (20), we get

(21) $0^{3k}(B1)1^n2^{4k} \vdash (28k^2 + (8n+15)k) \quad (B1)1^{7k+n}$.

By taking $n = 8$ in (21), we get
(22) $0^{3k}(B1)1^8 2^{4k} \quad \vdash (28k^2 + 79k) \quad (B1)1^{7k+8}$.

Using (8), (14) and (15), we get
(23) $2^{2k+1}(B1)0^2 \quad \vdash (8k + 9) \quad (A1)2^{2k+2}0$.

We are now ready to prove the results of the theorem.

Using (2), (22) and (16), we get
(24) $0^{3k+3}(A0)2^{4k+5} \quad \vdash (28k^2 + 86k + 61) \quad 2^{7k+8}(B1)$.

Using (24), (23) and (5) we get
(25) $0^{6k+4}(A0)2^{8k+5}0^2 \quad \vdash (112k^2 + 228k + 97) \quad 1(H1)2^{14k+9}0$.

Using (24), (23) and (3) we get
$0^{6k+7}(A0)2^{8k+9}0^2 \quad \vdash (112k^2 + 340k + 241) \quad (A0)2^{14k+17}0$,
and this result is still true for $k = -1$, so we have
(26) $0^{6k+1}(A0)2^{8k+1}0^2 \quad \vdash (112k^2 + 116k + 13) \quad (A0)2^{14k+3}0$.

Using (2), (22), (19) and (16) we get
(27) $0^{3k+6}(A0)2^{4k+6} \quad \vdash (28k^2 + 100k + 94) \quad 1^3 2^{7k+9}(B1)$.

Using (27), (23), (14) and (3) we get
(28) $0^{6k+7}(A0)2^{8k+6}0^2 \quad \vdash (112k^2 + 256k + 139) \quad (A0)2^{14k+14}0$.

Using (27), (23), (6), (14) and (3) we get
$0^{6k+10}(A0)2^{8k+10}0^2 \quad \vdash (112k^2 + 368k + 294) \quad (A0)2^{14k+21}0$,
and this result is still true for $k = -1$, so we have
(29) $0^{6k+4}(A0)2^{8k+2}0^2 \quad \vdash (112k^2 + 144k + 38) \quad (A0)2^{14k+7}0$.

Using (2), (22), (19), (18), (16), (8) and (14) we get
(30) $0^{3k+6}(A0)2^{4k+7}0^2 \quad \vdash (28k^2 + 114k + 126) \quad 1^2 2^{7k+10}(A1)220$.

Using (30), (15), (14) and (3) we get
(31) $0^{6k+7}(A0)2^{8k+7}0^2 \quad \vdash (112k^2 + 284k + 169) \quad (A0)2^{14k+15}0$.

Using (30), (15), (6), (4) and (3) we get
$0^{6k+10}(A0)2^{8k+11}0^2 \quad \vdash (112k^2 + 396k + 338) \quad (A0)2^{14k+22}0$,
and this result is still true for $k = -1$, so we have
(32) $0^{6k+4}(A0)2^{8k+3}0^2 \quad \vdash (112k^2 + 172k + 54) \quad (A0)2^{14k+8}0$.

Using (2), (22), (19), (18), (18), (16), (8) and (14) we get
(33) $0^{3k+6}(A0)2^{4k+8}0^2 \quad \vdash (28k^2 + 128k + 153) \quad 12^{7k+12}(A1)220$.

Using (33), (15), (4) and (3) we get
(34) $0^{6k+7}(A0)2^{8k+8}0^2 \quad \vdash (112k^2 + 312k + 203) \quad (A0)2^{14k+16}0$.

Using (33), (15), (6) and (3) we get
$0^{6k+10}(A0)2^{8k+12}0^2 \quad \vdash (112k^2 + 424k + 386) \quad (A0)2^{14k+23}0$,
and this result is still true for $k = -1$, so we have
(35) $0^{6k+4}(A0)2^{8k+4}0^2 \quad \vdash (112k^2 + 200k + 74) \quad (A0)2^{14k+9}0$.

The results (1), (26), (29), (32), (35), (25), (28), (31) and (34) give, respectively, the results (a)–(i) of the theorem. □

Using the rules of this theorem, we have, in 34 transitions,

$$^{\omega}0(A0)0^{\omega} \vdash (3) \quad C(1) \vdash (13) \quad C(3) \vdash (\ ) \cdots \vdash (\ ) \quad ^{\omega}01(H1)2^{374,676,381}0^{\omega}.$$

M_2	0	1	2	3
A	1RB	2LA	1RA	1RA
B	1LB	1LA	3RB	1RH

Table 7: Machine M_2 discovered in February 2005 by T. and S. Ligocki

Let g_1 be the pure Collatz-like function defined by: for $k \geq 0$,

$$\begin{array}{rcl} g_1(8k+1) & = & 14k+3, \\ g_1(8k+2) & = & 14k+7, \\ g_1(8k+3) & = & 14k+8, \\ g_1(8k+4) & = & 14k+9, \\ g_1(8k+5) & & \text{undefined}, \\ g_1(8k+6) & = & 14k+14, \\ g_1(8k+7) & = & 14k+15, \\ g_1(8k+8) & = & 14k+16. \end{array}$$

Then $g_1^{33}(1)$ is undefined.

The theorem gives immediately the following proposition.

Proposition 3.2 *The behavior of Turing machine M_1, on inputs 02^n, $n \geq 1$, depends on the behavior of iterated $g_1^k(n)$, $k \geq 1$.*

Since the behavior of iterated $g_1^k(n)$ is an open problem in mathematics, the halting problem for Turing machine M_1 is so.

Let $h_1(n) = \min\{k : g_1^k(n) \text{ is undefined}\}$. We have seen that $h_1(1) = 33$. We also have $h_1(144) = 41$, $h_1(270) = 51$.

4 Collatz-like with parameter: first example

Let M_2 be the 2×4 Turing machine defined by Table 7.

We have $s(M_2) = 3{,}932{,}964$ and $\sigma(M_2) = 2050$.

This machine is the current champion for the busy beaver competition for 2×4 machines. It was discovered in February 2005 by Terry and Shawn Ligocki, who wrote (email on February, 13th) that they found this machine using simulated annealing.

The following theorem gives the rules that enable Turing machine M_2 to reach a halting configuration from a blank tape.

Theorem 4.1 *Let*

$$\begin{array}{rcl} C(n,1) & = & {}^{\omega}0(A0)2^n10^{\omega}, \\ C(n,2) & = & {}^{\omega}0(A0)2^n110^{\omega}. \end{array}$$

Then

(a) $\ {}^{\omega}0(A0)0^{\omega} \ \vdash (6) \ \ C(1,2)$,

and, for all $k \geq 0$,

(b) $C(3k,1)$ $\vdash (15k^2+9k+3)$ $C(5k+1,1)$,
(c) $C(3k+1,1)$ $\vdash (15k^2+24k+13)$ ${}^{\omega}013^{5k+2}1(H1)0^{\omega}$,
(d) $C(3k+2,1)$ $\vdash (15k^2+29k+17)$ $C(5k+4,2)$,
(e) $C(3k,2)$ $\vdash (15k^2+11k+3)$ $C(5k+1,2)$,
(f) $C(3k+1,2)$ $\vdash (15k^2+21k+7)$ $C(5k+3,1)$,
(g) $C(3k+2,2)$ $\vdash (15k^2+36k+23)$ ${}^{\omega}013^{5k+4}1(H1)0^{\omega}$.

Proof. A direct inspection of the transition table gives

(1) $0^2(A0)0 \vdash (6) \ (A0)211$,
(2) $1(A0)0 \vdash (3) \ (A1)11$,
(3) $(A0)2 \vdash (1) \ 1(B2)$,
(4) $0(A1) \vdash (1) \ (A0)2$,
(5) $1(A1) \vdash (1) \ (A1)2$,
(6) $3(A1) \vdash (2) \ 1(A2)$,
(7) $(A2)0 \vdash (1) \ 1(A0)$,
(8) $(A2)1 \vdash (1) \ 1(A1)$,
(9) $(A2)2 \vdash (1) \ 1(A2)$,
(10) $(B2)0 \vdash (3) \ 1(H1)$,
(11) $(B2)1 \vdash (4) \ (A1)2$,
(12) $(B2)2 \vdash (1) \ 3(B2)$.

Iterating, respectively, (5), (9) and (12) gives

(13) $1^k(A1) \vdash (k) \ (A1)2^k$,
(14) $(A2)2^k \vdash (k) \ 1^k(A2)$,
(15) $(B2)2^k \vdash (k) \ 3^k(B2)$.

Using (2), (13) and (4) we get

$01^{k+1}(A0)0 \vdash (k+4) \ (A0)2^{k+1}11$,

and this result is still true for $k=-1$, so we have

(16) $01^k(A0)0 \vdash (k+3) \ (A0)2^k11$.

Using (2), (13), (6), (14), (8), (13) and (4) we get

(17) $0131^{k+1}(A0)0 \vdash (3k+10) \ (A0)2^{k+4}1$.

Using (2), (13), (6), (14), (8), (13), (6), (14), (8), (13) and (4) we get

(18) $01331^{k+1}(A0)0 \vdash (5k+19) \ (A0)2^{k+6}$.

Using (3), (15) and (10) we get

(19) $(A0)2^{k+1}0 \vdash (k+4) \ 13^k1(H1)$.

Using (3), (15), (11), (6), (9) and (7) we get

(20) $(A0)2^{k+2}10 \vdash (k+10) \ 13^k1^3(A0)$.

Using (3), (15), (11), (6), (9), (8), (13), (6), (14) and (7) we get

(21) $(A0)2^{k+3}110 \vdash (k+20) \ 13^k1^5(A0)$.

Using (2), (13), (6), (14), (8), (13), (6), (14), (8), (13), (6), (14) and (7) we get

(22) $3^31^{k+1}(A0)0^2 \vdash (6k+24) \ 1^{k+6}(A0)$.

By induction on k, from (22), we get

(23) $3^{3k}1^{n+1}(A0)0^{2k} \vdash (15k^2+6nk+9k) \ 1^{5k+n+1}(A0)$,

so we have, for $n=2$ and $n=4$

(24) $3^{3k}1^3(A0)0^{2k} \quad \vdash (15k^2+21k) \quad 1^{5k+3}(A0)$,
(25) $3^{3k}1^5(A0)0^{2k} \quad \vdash (15k^2+33k) \quad 1^{5k+5}(A0)$.

We are now ready to prove the theorem.
Using (20), (24) and (17) we get

$0(A0)2^{3k+3}10^{2k+2} \quad \vdash (15k^2+39k+27) \quad (A0)2^{5k+6}1$,

and the result is still true for $k=-1$, so we have

(26) $0(A0)2^{3k}10^{2k} \quad \vdash (15k^2+9k+3) \quad (A0)2^{5k+1}1$.

Using (20), (24), (18) and (19) we get

$0(A0)2^{3k+4}10^{2k+3} \quad \vdash (15k^2+54k+52) \quad 13^{5k+7}1(H1)$,

and the result is still true for $k=-1$, so we have

(27) $0(A0)2^{3k+1}10^{2k+1} \quad \vdash (15k^2+24k+13) \quad 13^{5k+2}1(H1)$.

Using (20), (24) and (16) we get

(28) $0(A0)2^{3k+2}10^{2k+2} \quad \vdash (15k^2+29k+17) \quad (A0)2^{5k+4}11$.

Using (21), (25) and (16) we get

$0(A0)2^{3k+3}110^{2k+2} \quad \vdash (15k^2+41k+29) \quad (A0)2^{5k+6}11$,

and the result is still true for $k=-1$, so we have

(29) $0(A0)2^{3k}110^{2k} \quad \vdash (15k^2+11k+3) \quad (A0)2^{5k+1}11$.

Using (21), (25) and (17) we get

$0(A0)2^{3k+4}110^{2k+2} \quad \vdash (15k^2+51k+43) \quad (A0)2^{5k+8}1$,

and the result is still true for $k=-1$, so we have

(30) $0(A0)2^{3k+1}110^{2k} \quad \vdash (15k^2+21k+7) \quad (A0)2^{5k+3}1$.

Using (21), (25), (18) and (19) we get

$0(A0)2^{3k+5}110^{2k+3} \quad \vdash (15k^2+66k+74) \quad 13^{5k+9}1(H1)$,

and the result is still true for $k=-1$, so we have

(31) $0(A0)2^{3k+2}110^{2k+1} \quad \vdash (15k^2+36k+23) \quad 13^{5k+4}1(H1)$.

The theorem comes from results (1) and (26)–(31). $\square$

Using the rules of this theorem, we have, in 14 transitions,

$$^{\omega}0(A0)0^{\omega} \vdash (6) \quad C(1,2) \vdash (7) \quad C(3,1) \vdash (\ ) \cdots \vdash (\ ) \quad ^{\omega}013^{2047}1(H1)0^{\omega}.$$

Let g_2 be the pure Collatz-like function with parameter defined by: for $k \geq 0$,

$$\begin{array}{rcl} g_2(3k,1) & = & (5k+1,1), \\ g_2(3k+1,1) & & \text{undefined}, \\ g_2(3k+2,1) & = & (5k+4,2), \\ g_2(3k,2) & = & (5k+1,2), \\ g_2(3k+1,2) & = & (5k+3,1), \\ g_2(3k+2,2) & & \text{undefined}. \end{array}$$

Then $g_2^{13}(1,2)$ is undefined.
The theorem gives immediately the following proposition.

Proposition 4.2 *The behavior of Turing machine M_2, on inputs 02^n1^i, $n \geq 1$, $i \in \{1,2\}$ depends on the behavior of iterated $g_2^k(n,i)$, $k \geq 1$.*

M_3	0	1	2	3	4
A	1RB	2LA	1RA	2LB	2LA
B	0LA	2RB	3RB	4RA	1RH

Table 8: Machine M_3 discovered in November 2007 by T. and S. Ligocki

Since the behavior of iterated $g_2^k(n,i)$ is an open problem in mathematics, the halting problem for Turing machine M_2 is so.

Let $h_2(n,i) = \min\{k : g_2^k(n,i) \text{ is undefined}\}$. We have seen that $h_2(1,2) = 13$. We also have $h_2(137,1) = 16$, $h_2(210,2) = 20$.

5 Collatz-like with parameter: second example

Let M_3 be the 2×5 Turing machine defined by Table 8.

We have $s(M_3) > 1.9 \times 10^{704}$ and $\sigma(M_3) > 1.7 \times 10^{352}$.

This machine is the current champion for the busy beaver competition for 2×5 machines. It was discovered in November 2007 by Terry and Shawn Ligocki, who wrote (email on November, 9th) that, as they did for 3×3 machine M_1, they enumerated all the 2×5 machines and applied the techniques of acceleration and proof systems originally developed by Marxen and Buntrock.

The following theorem gives the rules that enable Turing machine M_3 to reach a halting configuration from a blank tape.

Theorem 5.1 *Let*

$$\begin{array}{lcl}
C(n,1) & = & {}^\omega 013^n(B0)0^\omega, \\
C(n,2) & = & {}^\omega 023^n(B0)0^\omega, \\
C(n,3) & = & {}^\omega 03^n(B0)0^\omega, \\
C(n,4) & = & {}^\omega 04113^n(B0)0^\omega, \\
C(n,5) & = & {}^\omega 04123^n(B0)0^\omega, \\
C(n,6) & = & {}^\omega 0413^n(B0)0^\omega, \\
C(n,7) & = & {}^\omega 0423^n(B0)0^\omega, \\
C(n,8) & = & {}^\omega 043^n(B0)0^\omega.
\end{array}$$

Then

$(a) \quad {}^\omega 0(A0)0^\omega \quad \vdash (1) \quad C(0,1),$

and, for all $k \geq 0$,

$$\begin{array}{llll}
(b) & C(2k,1) & \vdash (3k^2+8k+4) & C(3k+1,1),\\
(c) & C(2k,2) & \vdash (3k^2+14k+9) & C(3k+2,1),\\
(d) & C(2k,3) & \vdash (3k^2+8k+2) & C(3k,1),\\
(e) & C(2k,4) & \vdash (3k^2+8k+8) & C(3k+3,1),\\
(f) & C(2k,5) & \vdash (3k^2+14k+13) & C(3k+4,1),\\
(g) & C(2k,6) & \vdash (3k^2+8k+6) & C(3k+2,1),\\
(h) & C(2k,7) & \vdash (3k^2+14k+11) & C(3k+3,1),\\
(i) & C(2k,8) & \vdash (3k^2+8k+4) & C(3k+1,1),\\
(j) & C(2k+1,1) & \vdash (3k^2+8k+4) & C(3k+1,2),\\
(k) & C(2k+1,2) & \vdash (3k^2+8k+4) & C(3k+2,3),\\
(l) & C(2k+1,3) & \vdash (3k^2+8k+22) & C(3k+1,4),\\
(m) & C(2k+1,4) & \vdash (3k^2+8k+4) & C(3k+1,5),\\
(n) & C(2k+1,5) & \vdash (3k^2+8k+4) & C(3k+2,6),\\
(o) & C(2k+1,6) & \vdash (3k^2+8k+4) & C(3k+1,7),\\
(p) & C(2k+1,7) & \vdash (3k^2+8k+4) & C(3k+2,8),\\
(q) & C(2k+1,8) & \vdash (3k^2+5k+3) & {}^{\omega}01(H2)2^{3k}0^{\omega}.
\end{array}$$

Proof. A direct inspection of the transition table gives

$$\begin{array}{llll}
(1) & (A0)0 & \vdash (1) & 1(B0),\\
(2) & 0(A0)0 & \vdash (17) & 41(A0),\\
(3) & (A0)2 & \vdash (1) & 1(B2),\\
(4) & 0(A1) & \vdash (1) & (A0)2,\\
(5) & 1(A1) & \vdash (1) & (A1)2,\\
(6) & 4(A1) & \vdash (1) & (A4)2,\\
(7) & (A2)0^2 & \vdash (2) & 1^2(B0),\\
(8) & (A2)2 & \vdash (1) & 1(A2),\\
(9) & 1(B0) & \vdash (1) & (A1)0,\\
(10) & 3^2(B0)0 & \vdash (5) & 41^2(B0),\\
(11) & (B2)0 & \vdash (1) & 3(B0),\\
(12) & (B2)2 & \vdash (1) & 3(B2),\\
(13) & 0(A4) & \vdash (1) & (A0)2,\\
(14) & 1(A4) & \vdash (1) & (A1)2,\\
(15) & 2(A4) & \vdash (1) & (A2)2,\\
(16) & 0^23(A4) & \vdash (3) & (A0)02^2,\\
(17) & 13(A4) & \vdash (4) & 23(B2),\\
(18) & 23(A4) & \vdash (4) & 33(B2),\\
(19) & 3^2(A4) & \vdash (4) & 41(A2),\\
(20) & 43(A4) & \vdash (3) & 1(H2)2,\\
(21) & 04(A4) & \vdash (2) & (A0)2^2.
\end{array}$$

Iterating, respectively, (5), (8) and (12) gives

$$\begin{array}{llll}
(22) & 1^k(A1) & \vdash (k) & (A1)2^k,\\
(23) & (A2)2^k & \vdash (k) & 1^k(A2),\\
(24) & (B2)2^k & \vdash (k) & 3^k(B2).
\end{array}$$

Using (9), (22), (6), (19), (23) and (7) we get

$$3^241^{k+1}(B0)0 \quad \vdash (2k+9) \quad 41^{k+4}(B0),$$

and the result is still true for $k=-1$, so we have

(25) $3^2 41^k(B0)0 \quad \vdash (2k+7) \quad 41^{k+3}(B0)$.

For any $n \geq 0$, by induction on k, using (25), we get

(26) $3^{2k}41^n(B0)0^k \quad \vdash (3k^2+(2n+4)k) \quad 41^{3k+n}(B0)$,

so we have, for $n=2$ in (26),

(27) $3^{2k}41^2(B0)0^k \quad \vdash (3k^2+8k) \quad 41^{3k+2}(B0)$.

Using (10), (27), (9), (22) and (6), we get

(28) $3^{2k+2}(B0)0^{k+1} \quad \vdash (3k^2+11k+8) \quad (A4)2^{3k+2}0$.

Using (3), (24) and (11) we get

$(A0)2^{k+1}0 \quad \vdash (k+2) \quad 13^{k+1}(B0)$,

and the result is still true for $k=-1$, so we have

(29) $(A0)2^k0 \quad \vdash (k+1) \quad 13^k(B0)$.

Using (15), (23), (7), (9) and (22), we get

(30) $2(A4)2^k0^2 \quad \vdash (2k+7) \quad (A1)2^{k+2}0$.

We are now ready to prove the theorem.

Using (28), (14), (4) and (29) we get

$013^{2k+2}(B0)0^{k+1} \quad \vdash (3k^2+14k+15) \quad 13^{3k+4}(B0)$,

and the result is still true for $k=-1$, so we have

(31) $013^{2k}(B0)0^k \quad \vdash (3k^2+8k+4) \quad 13^{3k+1}(B0)$.

Using (28), (30), (4) and (29) we get

$023^{2k+2}(B0)0^{k+2} \quad \vdash (3k^2+20k+26) \quad 13^{3k+5}(B0)$,

and the result is still true for $k=-1$, so we have

(32) $023^{2k}(B0)0^{k+1} \quad \vdash (3k^2+14k+9) \quad 13^{3k+2}(B0)$.

Using (28), (13) and (29) we get

$03^{2k+2}(B0)0^{k+1} \quad \vdash (3k^2+14k+13) \quad 13^{3k+3}(B0)$,

and the result is still true for $k=-1$, so we have

(33) $03^{2k}(B0)0^k \quad \vdash (3k^2+8k+2) \quad 13^{3k}(B0)$.

Using (28), (14), (5), (6), (13) and (29) we get

$04113^{2k+2}(B0)0^{k+1} \quad \vdash (3k^2+14k+19) \quad 13^{3k+6}(B0)$,

and the result is still true for $k=-1$, so we have

(34) $04113^{2k}(B0)0^k \quad \vdash (3k^2+8k+8) \quad 13^{3k+3}(B0)$.

Using (28), (30), (5), (6), (13) and (29) we get

$04123^{2k+2}(B0)0^{k+2} \quad \vdash (3k^2+20k+30) \quad 13^{3k+7}(B0)$,

and the result is still true for $k=-1$, so we have

(35) $04123^{2k}(B0)0^{k+1} \quad \vdash (3k^2+14k+13) \quad 13^{3k+4}(B0)$.

Using (28), (14), (6), (13) and (29) we get

$0413^{2k+2}(B0)0^{k+1} \quad \vdash (3k^2+14k+17) \quad 13^{3k+5}(B0)$,

and the result is still true for $k=-1$, so we have

(36) $0413^{2k}(B0)0^k \quad \vdash (3k^2+8k+6) \quad 13^{3k+2}(B0)$.

Using (28), (30), (6), (13) and (29) we get

$0423^{2k+2}(B0)0^{k+2} \quad \vdash (3k^2+20k+28) \quad 13^{3k+6}(B0)$,

and the result is still true for $k=-1$, so we have

(37) $0423^{2k}(B0)0^{k+1} \quad \vdash (3k^2+14k+11) \quad 13^{3k+3}(B0)$.

Using (28), (21) and (29) we get

$043^{2k+2}(B0)0^{k+1} \quad \vdash (3k^2+14k+15) \quad 13^{3k+4}(B0),$

and the result is still true for $k=-1$, so we have

(38) $\quad 043^{2k}(B0)0^k \quad \vdash (3k^2+8k+4) \quad 13^{3k+1}(B0).$

Using (28), (17), (24) and (11) we get

$13^{2k+3}(B0)0^{k+1} \quad \vdash (3k^2+14k+15) \quad 23^{3k+4}(B0),$

and the result is still true for $k=-1$, so we have

(39) $\quad 13^{2k+1}(B0)0^k \quad \vdash (3k^2+8k+4) \quad 23^{3k+1}(B0).$

Using (28), (18), (24) and (11) we get

$23^{2k+3}(B0)0^{k+1} \quad \vdash (3k^2+14k+15) \quad 3^{3k+5}(B0),$

and the result is still true for $k=-1$, so we have

(40) $\quad 23^{2k+1}(B0)0^k \quad \vdash (3k^2+8k+4) \quad 3^{3k+2}(B0).$

Using (28), (16), (2) and (29) we get

$0^33^{2k+3}(B0)0^{k+1} \quad \vdash (3k^2+14k+33) \quad 41^23^{3k+4}(B0),$

and the result is still true for $k=-1$, so we have

(41) $\quad 0^33^{2k+1}(B0)0^k \quad \vdash (3k^2+8k+22) \quad 41^23^{3k+1}(B0).$

Using (39) we get

(42) $\quad 04113^{2k+1}(B0)0^k \quad \vdash (3k^2+8k+4) \quad 04123^{3k+1}(B0).$

Using (40) we get

(43) $\quad 04123^{2k+1}(B0)0^k \quad \vdash (3k^2+8k+4) \quad 0413^{3k+2}(B0).$

Using (39) we get

(44) $\quad 0413^{2k+1}(B0)0^k \quad \vdash (3k^2+8k+4) \quad 0423^{3k+1}(B0).$

Using (40) we get

(45) $\quad 0423^{2k+1}(B0)0^k \quad \vdash (3k^2+8k+4) \quad 043^{3k+2}(B0).$

Using (28) and (20) we get

$43^{2k+3}(B0)0^{k+1} \quad \vdash (3k^2+11k+11) \quad 1(H2)2^{3k+3}0,$

and the result is still true for $k=-1$, so we have

(46) $\quad 43^{2k+1}(B0)0^k \quad \vdash (3k^2+5k+3) \quad 1(H2)2^{3k}0.$

Results (1) and (31)–(46) give results (a)–(p) of the theorem. □

Using the rules of this theorem, we have, in 2002 transitions,

$$^{\omega}0(A0)0^{\omega} \vdash (1) \quad C(0,1) \vdash (4) \quad C(1,1) \vdash (\ ) \cdots \vdash (\ ) \quad \text{END}.$$

Let g_3 be the pure Collatz-like function with parameter defined by: for $k \geq 0$,

$g_3(2k,1)$	$=$	$(3k+1,1)$	$g_3(2k+1,1)$	$=$	$(3k+1,2)$
$g_3(2k,2)$	$=$	$(3k+2,1)$	$g_3(2k+1,2)$	$=$	$(3k+2,3)$
$g_3(2k,3)$	$=$	$(3k,1)$	$g_3(2k+1,3)$	$=$	$(3k+1,4)$
$g_3(2k,4)$	$=$	$(3k+3,1)$	$g_3(2k+1,4)$	$=$	$(3k+1,5)$
$g_3(2k,5)$	$=$	$(3k+4,1)$	$g_3(2k+1,5)$	$=$	$(3k+2,6)$
$g_3(2k,6)$	$=$	$(3k+2,1)$	$g_3(2k+1,6)$	$=$	$(3k+1,7)$
$g_3(2k,7)$	$=$	$(3k+3,1)$	$g_3(2k+1,7)$	$=$	$(3k+2,8)$
$g_3(2k,8)$	$=$	$(3k+1,1)$	$g_3(2k+1,8)$		undefined

Then $g_3^{2001}(0,1)$ is undefined.

M_4	0	1
A	1RB	1LE
B	1RC	1RF
C	1LD	0RB
D	1RE	0LC
E	1LA	0RD
F	1RH	1RC

Table 9: Machine M_4 discovered in June 2010 by P. Kropitz

Proposition 5.2 *The behavior of Turing machine M_3, on inputs 02^n, $n \geq 1$, depends on the behavior of iterated $g_3^k(n, 1)$, $k \geq 1$.*

Proof. We have ${}^\omega 0(A0)2^n 0^\omega \vdash (n+1) \quad {}^\omega 013^n(B0)0^\omega = C(n, 1)$. □

Since the behavior of iterated $g_3^k(n, 1)$ is an open problem in mathematics, the halting problem for Turing machine M_3 is so.

Note that the way by which a high score is obtained is particularly clear for machine M_3. The parameter p, $1 \leq p \leq 8$, can be seen as a state. If n is odd, $g_3(n, p) = (n', p+1)$, the state goes from p to $p+1$, and the computation stops when state 8 is reached. If n is even, $g_3(n, p) = (n', 1)$, and the state goes back to 1.

6 Exponential Collatz-like

Let M_4 be the 6×2 Turing machine defined by Table 9.

We have $s(M_4) > 7.4 \times 10^{36534}$ and $\sigma(M_4) > 3.5 \times 10^{18267}$.

This machine is the current champion for the busy beaver competition for 6×2 machines. It was discovered in June 2010 by Pavel Kropitz.

The following theorem gives the rules observed by Turing machine M_4.

Theorem 6.1 *Let $C(n) = {}^\omega 0(A0)1^n 0^\omega$. Then*

(a) $C(0) \vdash (29) \quad C(9)$,
(b) $C(2) \vdash (36) \quad C(11)$,
(c) $C(3) \vdash (48) \quad C(13)$,

and, for all $k \geq 0$,

(d) $C(3k+1) \vdash (3k+3) \quad {}^\omega 0111(011)^k(H0)0^\omega$,
(e) $C(9k+5) \vdash ((4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 25362)/270) \quad C((98 \times 4^k - 11)/3)$,
(f) $C(9k+6) \vdash ((125 \times 16^{k+2} - 575 \times 4^{k+2} + 228k - 2226)/27) \quad C((50 \times 4^{k+1} - 59)/3)$,
(g) $C(9k+8) \vdash ((4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 11592)/270) \quad C((98 \times 4^k + 1)/3)$,
(h) $C(9k+9) \vdash ((125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 2289)/27) \quad C((50 \times 4^{k+1} - 11)/3)$,
(i) $C(9k+11) \vdash ((4802 \times 16^{k+2} - 11270 \times 4^{k+2} + 2280k - 22452)/270) \quad C((98 \times 4^{k+1} - 59)/3)$,
(j) $C(9k+12) \vdash ((125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 912)/27) \quad C((50 \times 4^{k+1} + 1)/3)$.

Note that the behavior of this Turing machine on the blank tape involves only items (a), (d), (h) and (j).

Proof. A direct inspection of the transition table gives

(1)	$0^3(A0)0^6$	$\vdash (29)$	$(A0)1^9$,
(2)	$0^4(A0)1^2 0^5$	$\vdash (36)$	$(A0)1^{11}$,
(3)	$0^4(A0)1^3 0^6$	$\vdash (48)$	$(A0)1^{13}$,
(4)	$(A0)1$	$\vdash (1)$	$1(B1)$,
(5)	$01(E0)$	$\vdash (2)$	$(E0)11$,
(6)	$0(E0)$	$\vdash (1)$	$(A0)1$,
(7)	$11(E0)$	$\vdash (2)$	$(E1)11$,
(8)	$01(C0)$	$\vdash (2)$	$(C0)01$,
(9)	$11(C0)$	$\vdash (2)$	$(C1)01$,
(10)	$0(C0)$	$\vdash (2)$	$1(E1)$,
(11)	$(E1)01$	$\vdash (2)$	$01(E1)$,
(12)	$(E1)00$	$\vdash (2)$	$01(E0)$,
(13)	$(E1)1$	$\vdash (2)$	$(C0)0$,
(14)	$(B1)1^3$	$\vdash (3)$	$110(B1)$,
(15)	$(B1)00$	$\vdash (2)$	$11(H0)$,
(16)	$(B1)10$	$\vdash (6)$	$01(C1)$,
(17)	$(B1)1100$	$\vdash (12)$	$(01)^2(C1)$,
(18)	$(C1)01$	$\vdash (2)$	$01(C1)$,
(19)	$(C1)00$	$\vdash (2)$	$01(C0)$,
(20)	$(C1)1^6 0^6$	$\vdash (44)$	$01(E1)1^{10}$,
(21)	$(C1)1^8 0^{11}$	$\vdash (113)$	$1(01)^5(E1)1^8$.

Iterating, respectively, (5), (8), (11), (14) and (18) gives

(22)	$(01)^k(E0)$	$\vdash (2k)$	$(E0)1^{2k}$,
(23)	$(01)^k(C0)$	$\vdash (2k)$	$(C0)(01)^k$,
(24)	$(E1)(01)^k$	$\vdash (2k)$	$(01)^k(E1)$,
(25)	$(B1)1^{3k}$	$\vdash (3k)$	$(110)^k(B1)$,
(26)	$(C1)(01)^k$	$\vdash (2k)$	$(01)^k(C1)$.

Using (19), (23), (10) and (24), we get

(27) $0(01)^k(C1)00 \ \vdash (4k+8) \ \ 1(01)^{k+1}(E1)$.

Using (12), (22) and (6), we get

(28) $0(01)^k(E1)00 \ \vdash (2k+5) \ \ (A0)1^{2k+3}$.

Using (12), (22) and (7), we get

(29) $11(01)^k(E1)00 \ \vdash (2k+6) \ \ (E1)1^{2k+4}$.

Using (13), (23), (10) and (24), we get

(30) $0(01)^k(E1)1 \ \vdash (4k+4) \ \ 1(01)^k(E1)0$.

Using (13), (23), (9) and (26), we get

(31) $11(01)^k(E1)1 \ \vdash (4k+6) \ \ (01)^{k+1}(C1)0$.

Using (20), (30), (11), (31) and (18), we get

(32) $10(01)^k(C1)1^6 0^6 \ \vdash (8k+70) \ \ (01)^{k+4}(C1)1^6$.

By induction on n, using (32), we get

(33) $(10)^n(01)^k(C1)1^6 0^{6n} \ \vdash (16n^2+8kn+54n) \ \ (01)^{4n+k}(C1)1^6$.

Using (30) and (11), we get

(34) $00(01)^k(E1)11 \ \vdash (4k+6) \ \ (01)^{k+2}(E1)$.

By induction on n, using (34), we get

(35) $0^{2n}(01)^k(E1)1^{2n} \quad \vdash (4n^2+4kn+2n) \quad (01)^{2n+k}(E1)$.

Using (20), (31), (18), (21), (31), (18) and (33), we get

(36) $11(01)^k(C1)1^6 0^{6k+29} \quad \vdash (16k^2+178k+481) \quad 0(01)^{4k+15}(C1)1^6$.

Using (20), (35) and (28), we get

(37) $0^{11}(01)^k(C1)1^6 0^8 \quad \vdash (22k+201) \quad (A0)1^{2k+25}$.

Using (20), (35), (31), (18), (32) and (36), we get

(38) $(110)^3 0(01)^k(C1)1^6 0^{6k+101} \quad \vdash (16k^2+514k+4045) \quad 0(01)^{4k+55}(C1)1^6$.

By induction on k, using (38) we get

(39) $(110)^{3k}0(01)^n(C1)1^6 0^a \quad \vdash (T) \quad 0(01)^b(C1)1^6$,

with $a = (2(3n+55)4^k - 6n - 27k - 110)/3$, $b = ((3n+55)4^k - 55)/3$, and $T = \frac{16(3n+55)^2}{135}16^k - \frac{218(3n+55)}{27}4^k - \frac{5}{9}k - \frac{16(3n+55)^2}{135} + \frac{218(3n+55)}{27}$.

Using (4), (25), (17), (27), (29), (31), (18), (21), (31), (18) and (33), we get

(40) $(A0)1^{3k+9}0^{29} \quad \vdash (3k+484) \quad 1(110)^k 0(01)^{15}(C1)1^6$.

Using (40), (39), (32) and (37), we get

(41) $0^{11}(A0)1^{9k+9}0^a \quad \vdash (T) \quad (A0)1^b$,

with $a = (50\times 4^{k+1} - 11)/3 - 9k - 20$, $b = (50\times 4^{k+1} - 11)/3$, and $T = (125\times 16^{k+2} + 325\times 4^{k+2} + 228k - 2289)/27$.

Using (40), (39), (20), (35), (31), (18) and (37), we get

(42) $0^{12}(A0)1^{9k+12}0^a \quad \vdash (T) \quad (A0)1^b$,

with $a = (50\times 4^{k+1} + 1)/3 - 9k - 24$, $b = (50\times 4^{k+1} + 1)/3$, and $T = (125\times 16^{k+2} + 325\times 4^{k+2} + 228k - 912)/27$.

Using (4), (25) and (15), we get

(43) $(A0)1^{3k+1}00 \quad \vdash (3k+3) \quad 111(011)^k(H0)$.

Results (1), (43), (41) and (42) are results (a), (d), (h) and (j) of the theorem. They are sufficient to analyze the behavior of the Turing machine on a blank tape. The following gives its behavior from configurations ${}^\omega 0(A0)1^n 0^\omega$, where $n = 9k+m$, $m \in \{5, 6, 8, 11\}$.

Using (4), (25), (16), (27), (29), (31), (18), (32) and (36), we get

(44) $(A0)1^{3k+14}0^{88} \quad \vdash (3k+3076) \quad 1(110)^k 0(01)^{47}(C1)1^6$.

Using (44), (39), (32) and (37), we get

$0^{11}(A0)1^{9k+14}0^a \quad \vdash (T) \quad (A0)1^b$,

with $b = (98\times 4^{k+1} - 11)/3$ and $T = (4802\times 16^{k+2} + 6370\times 4^{k+2} + 2280(k+1) - 25362)/270$, and this result is still true for $k = -1$, since

$0^{11}(A0)1^5 0^{13} \quad \vdash (285) \quad (A0)1^{29}$,

so we get

(45) $0^{11}(A0)1^{9k+5}0^a \quad \vdash (T) \quad (A0)1^b$,

with $b = (98\times 4^k - 11)/3$ and $T = (4802\times 16^{k+1} + 6370\times 4^{k+1} + 2280k - 25362)/270$.

Using (20), (35), (31), (18), (32), (36) and (37), we get

(46) $0^{10}1(110)^2 0(01)^k(C1)1^6 0^{6k+103} \quad \vdash (16k^2+570k+4886) \quad (A0)1^{8k+127}$.

Using (40), (39) and (46), we get

$0^{10}(A0)1^{9k+15}0^a \quad \vdash (T) \quad (A0)1^b$, with $b = (50 \times 4^{k+2} - 59)/3$ and $T = (125 \times 16^{k+3} - 575 \times 4^{k+3} + 228(k+1) - 2226)/27$, and this result is still true for $k = -1$, since

$0^{10}(A0)1^6 0^{31} \quad \vdash (762) \quad (A0)1^{47}$,

so we get

(47) $\quad 0^{10}(A0)1^{9k+6}0^a \quad \vdash (T) \quad (A0)1^b$,

with $b = (50 \times 4^{k+1} - 59)/3$ and $T = (125 \times 16^{k+2} - 575 \times 4^{k+2} + 228k - 2226)/27$.

Using (20), (30), (11), (31), (18) and (37), we get

(48) $\quad 0^{12}11100(01)^k(C1)1^6 0^{14} \quad \vdash (30k + 407) \quad (A0)1^{2k+37}$.

Using (44), (39) and (48), we get

$0^{12}(A0)1^{9k+17}0^a \quad \vdash (T) \quad (A0)1^b$,

with $b = (98 \times 4^{k+1} + 1)/3$ and $T = (4802 \times 16^{k+2} + 6370 \times 4^{k+2} + 2280(k+1) - 11592)/270$, and this result is still true for $k = -1$, since

$0^{12}(A0)1^8 0^{13} \quad \vdash (336) \quad (A0)1^{33}$,

so we get

(49) $\quad 0^{12}(A0)1^{9k+8}0^a \quad \vdash (T) \quad (A0)1^b$,

with $b = (98 \times 4^k + 1)/3$ and $T = (4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 11592)/270$.

Using (44), (39) and (46), we get

$0^{10}(A0)1^{9k+20}0^a \quad \vdash (T) \quad (A0)1^b$,

with $b = (98 \times 4^{k+2} - 59)/3$ and $T = (4802 \times 16^{k+3} - 11270 \times 4^{k+3} + 2280(k+1) - 22452)/270$, and this result is still true for $k = -1$, since

$0^{10}(A0)1^{11}0^{90} \quad \vdash (3802) \quad (A0)1^{111}$,

so we get

(50) $\quad 0^{10}(A0)1^{9k+11}0^a \quad \vdash (T) \quad (A0)1^b$,

with $b = (98 \times 4^{k+1} - 59)/3$ and $T = (4802 \times 16^{k+2} - 11270 \times 4^{k+2} + 2280k - 22452)/270$.

Results (45), (47), (49) and (50) are results (e), (f), (g) and (i) of the theorem.

Results (1), (2) and (3) give special cases (a), (b) and (c) of the theorem. □

Using the rules of this theorem, we have,

$$^{\omega}0(A0)0^{\omega} \vdash (29) \quad C(9) \vdash (1293) \quad C(63) \vdash (19,884,896,677)$$

$$C(273063) \vdash (T_1) \quad C((50 \times 4^{30340} + 1)/3) \vdash (T_2) \quad ^{\omega}0111(011)^K(H0)0^{\omega},$$

with $T_1 = (125 \times 16^{30341} + 325 \times 4^{30341} + 6916380)/27$, $T_2 = (50 \times 4^{30340} + 7)/3$, $K = (50 \times 4^{30340} - 2)/9$.

The total time is $s(M_4) = (125 \times 16^{30341} + 1750 \times 4^{30340} + 15)/27 + 19,885,154,163$, and the final number of symbols 1 is $\sigma(M_4) = (25 \times 4^{30341} + 23)/9$.

Let g_4 be the exponential Collatz-like function defined by: for $k \geq 0$,

M_5	0	1
A	1RB	0LD
B	1RC	0RF
C	1LC	1LA
D	0LE	1RH
E	1LA	0RB
F	0RC	0RE

Table 10: Machine M_5 discovered in May 2010 by P. Kropitz

$$
\begin{aligned}
g_4(0) &= 9, \\
g_4(2) &= 11, \\
g_4(3) &= 13, \\
g_4(3k+1) & \quad \text{undefined}, \\
g_4(9k+5) &= (98 \times 4^k - 11)/3, \\
g_4(9k+6) &= (50 \times 4^{k+1} - 59)/3, \\
g_4(9k+8) &= (98 \times 4^k + 1)/3, \\
g_4(9k+9) &= (50 \times 4^{k+1} - 11)/3, \\
g_4(9k+11) &= (98 \times 4^{k+1} - 59)/3, \\
g_4(9k+12) &= (50 \times 4^{k+1} + 1)/3.
\end{aligned}
$$

Then $g_4^5(0)$ is undefined.

The theorem gives immediately the following proposition.

Proposition 6.2 *The behavior of Turing machine M_4, on inputs 01^n, $n \geq 0$, depends on the behavior of iterated $g_4^k(n)$, $k \geq 1$.*

Since the behavior of iterated $g_4^k(n)$ is an open problem in mathematics, the halting problem for Turing machine M_4 is so.

Let $h_4(n) = \min\{k : g_4^k(n) \text{ is undefined}\}$. We have seen that $h_4(0) = 5$. We also have $h_4(2) = 8$, and $C(2) \vdash (T)$ END with $T > 10^{10^{10^{10^{18641000}}}}$. We also have $h_4(36) = 15$.

7 Unclassifiable machine

Let M_5 be the 6×2 Turing machine defined by Table 10.

We have $s(M_5) > 3.8 \times 10^{21132}$ and $\sigma(M_5) > 3.1 \times 10^{10566}$.

This machine was discovered in May 2010 by Pavel Kropitz. It was the champion for the busy beaver competition for 6×2 machines from May to June 2010.

The following theorem is adapted from an analysis of S. Ligocki [24]. It gives the rules that enable Turing machine M_5 to reach a halting configuration from a blank tape.

Theorem 7.1 *Let $C(k,n) = {}^{\omega}010^n1(C1)1^{3k}0^{\omega}$. Then*

(a) $\;{}^{\omega}0(A0)0^{\omega} \quad \vdash (47) \quad C(2,5)$,

and, for all $k \geq 0$,

(b)	$C(k,0)$	$\vdash (3)$	${}^{\omega}01(H0)1^{3k+1}0^{\omega}$,
(c)	$C(k,1)$	$\vdash (3k+37)$	$C(2,3k+2)$,
(d)	$C(k,2)$	$\vdash (12k+44)$	$C(k+2,4)$,
(e)	$C(k,3)$	$\vdash (3k+57)$	$C(2,3k+8)$,
(f)	$C(k,n+4)$	$\vdash (27k^2+105k+112)$	$C(3k+5,n)$.

Proof. A direct inspection of the transition table gives

(1)	$0^4(A0)0^9$	$\vdash (47)$	$10^51(C1)1^6$,
(2)	$0(C0)$	$\vdash (1)$	$(C0)1$,
(3)	$1(C0)$	$\vdash (1)$	$(C1)1$,
(4)	$(B1)00$	$\vdash (2)$	$0^2(C0)$,
(5)	$(B1)01$	$\vdash (4)$	$01(B1)$,
(6)	$(B1)10$	$\vdash (4)$	$01(B1)$,
(7)	$(B1)10^3$	$\vdash (14)$	$10^3(B1)$,
(8)	$(B1)1^20^2$	$\vdash (7)$	$0^21(B1)1$,
(9)	$(B1)1^3$	$\vdash (3)$	$0^3(B1)$,
(10)	$0(C1)$	$\vdash (2)$	$1(B1)$,
(11)	$11(C1)$	$\vdash (3)$	$1(H0)1$,
(12)	$0^31(C1)$	$\vdash (10)$	$(C1)1^4$,
(13)	$0^310^21(C1)$	$\vdash (8)$	$1(B1)0^210^21$,
(14)	$101(C1)$	$\vdash (8)$	$1(B1)1^2$.

Iterating, respectively, (2) and (9) gives

(15) $0^k(C0) \quad \vdash (k) \quad (C0)1^k$,

(16) $(B1)1^{3k} \quad \vdash (3k) \quad 0^{3k}(B1)$.

Using (4), (15) and (3), we get

(17) $10^k(B1)00 \quad \vdash (k+5) \quad (C1)1^{k+3}$.

Using (16), (17) and (10), we get

(18) $01(B1)1^{3k}00 \quad \vdash (6k+7) \quad 1(B1)1^{3k+3}$.

By induction on k, using (18), we get

(19) $0^k1(B1)0^{2k} \quad \vdash (3k^2+4k) \quad 1(B1)1^{3k}$.

Using (12), (10), (16), (6), (19), (16) and (17), we get

(20) $0^41(C1)1^{3k}0^{6k+11} \quad \vdash (27k^2+105k+112) \quad 1(C1)1^{9k+15}$.

Using (14), (16), (8), (7) and (17), we get

(21) $101(C1)1^{3k}0^7 \quad \vdash (3k+37) \quad 10^{3k+2}1(C1)1^6$.

Using (13), (17), (10), (9), (6), (5), (16), (17), (10), (16) and (17), we get

(22) $0^410^21(C1)1^{3k}0^4 \quad \vdash (12k+44) \quad 10^41(C1)1^{3k+6}$.

Using (12), (12), (10), (16), (8), (6), (17), (10), (9) and (17), we get

(23) $0^410^31(C1)1^{3k}0^7 \quad \vdash (3k+57) \quad 10^{3k+8}1(C1)1^6$.

Results (1), (11), (21), (22), (23) and (20) give results (a)–(f) of the theorem. □

Using the rules of this theorem, we have, in 22158 transitions,

$$ {}^{\omega}0(A0)0^{\omega} \vdash (47) \quad C(2,5) \vdash (430) \quad C(11,1) \vdash (\ ) \cdots \vdash (\ ) \quad \text{END}. $$

Let g_5 be the partial function defined by: for $k,n \geq 0$,

$$\begin{array}{rcl} g_5(k,0) & & \text{undefined}, \\ g_5(k,1) & = & (2,3k+2), \\ g_5(k,2) & = & (k+2,4), \\ g_5(k,3) & = & (2,3k+8), \\ g_5(k,n+4) & = & (3k+5,n). \end{array}$$

Then $g_5^{22157}(2,5)$ is undefined.

Proposition 7.2 *The behavior of Turing machine M_5, on inputs 01^{3n+3}, $n \geq 0$, depends on the behavior of iterated $g_5^k(n,p)$, $k \geq 1$, $n,p \geq 0$.*

Proof. We have ${}^\omega 0(A0)1^{3n+3}0^\omega \vdash (3n+30) \quad {}^\omega 01^{3n+2}1(C1)1^6 0^\omega = C(2,3n+2)$. □

Since the behavior of iterated $g_5^k(n,p)$ is an open problem in mathematics, the halting problem for Turing machine M_5 is so.

The following proposition shows that some configurations take a long time to halt.

Proposition 7.3 $C(9,1) \vdash (T)$ *END with* $T > 10^{10^{10^{10^{10^{3520}}}}}$.

Proof. By induction on n, using Theorem 7.1 (f), it is easy to prove that, if $n \geq 0$, $0 \leq r \leq 3$, we have

$$C(2,4n+r) \vdash (t_n) \quad C(u_n,r),$$

with $u_n = (3^{n+2}-5)/2$ and $t_n = (3 \times 9^{n+3} - 80 \times 3^{n+3} + 584n - 27)/32$.

By induction on k, it is easy to prove that, if $k \geq 2$, we have

$$3^{2^{k-1}} \equiv 2^{k+1} + 1 \pmod{2^{k+2}}$$

so the multiplicative order of 3 modulo 2^{k+2} is 2^k for $k \geq 1$. Thus we can prove that, for $k \geq 1$, $n,m \geq 0$, we have

$$n \equiv m \pmod{2^k} \iff u_n \equiv u_m \pmod{2^{k+1}}.$$

Now, suppose that, for $a \in \{1,3\}$, $n,n' \geq 1$, $q,q' \geq 1$, $0 \leq r,r' \leq 3$, we have

$$C(n,a) \vdash (3n+27+10a) \quad C(2,3n+3a-1) = C(2,4q+r) \vdash (t_q) \quad C(u_q,r),$$

and

$$C(n',a) \vdash (3n'+27+10a) \quad C(2,3n'+3a-1) = C(2,4q'+r') \vdash (t_{q'}) \quad C(u_{q'},r'),$$

and let $k \geq 2$ such that $n \equiv n' \pmod{2^{k+1}}$. Then it is easy to prove that $r = r'$ and $u_q \equiv u_{q'} \pmod{2^k}$. So the behavior of configurations $C(n,a)$ is mirrored by the behavior of configurations $C(n',a)$ with $n' \leq 2^k$ for suitable k.

In the following computation on $C(9,1)$:

$$\begin{array}{llll} C(9,1) & \vdash (\ ) & C(2,4\times 7+1) & \vdash (t_7) \\ C(9839,1) & \vdash (\ ) & C(2,4\times 7379+3) & \vdash (t_{7379}) \\ C(u_{7379},3) & \vdash (\ ) & C(2,4\times q_3+r_3) & \vdash (t_{q_3}) \\ C(u_{q_3},r_3) & \vdash (\ ) & C(2,4\times q_4+r_4) & \vdash (t_{q_4}) \\ C(u_{q_4},r_4) & \vdash (\ ) & C(2,4\times q_5+r_5) & \vdash (t_{q_5}) \\ C(u_{q_5},r_5) & \vdash (\ ) & C(2,4\times q_6+r_6) & \vdash (t_{q_6}) \\ C(u_{q_6},r_6) & \vdash (3) & \text{END} & \end{array}$$

M_6	0	1
A	1RB	0RF
B	0LB	1LC
C	1LD	0RC
D	1LE	1RH
E	1LF	0LD
F	1RA	0LE

Table 11: Machine M_6 discovered in November 2007 by T. and S. Ligocki

we know that $r_6 = 0$ because we have

$$\begin{array}{llll}
u_{q_1} = u_7 \equiv 47 \pmod{64}, & (3\times 47)+2 = (4\times 35)+3, & q'_2 = 35, & r_2 = 3,\\
u_{q_2} \equiv u_{q'_2} \equiv 23 \pmod{32}, & (3\times 23)+8 = (4\times 19)+1, & q'_3 = 19, & r_3 = 1,\\
u_{q_3} \equiv u_{q'_3} \equiv 7 \pmod{16}, & (3\times 7)+2 = (4\times 5)+3, & q'_4 = 5, & r_4 = 3,\\
u_{q_4} \equiv u_{q'_4} \equiv 3 \pmod{8}, & (3\times 3)+8 = (4\times 4)+1, & q'_5 = 4, & r_5 = 1,\\
u_{q_5} \equiv u_{q'_5} \equiv 2 \pmod{4}, & (3\times 2)+2 = (4\times 2)+0, & q'_6 = 2, & r_6 = 0.
\end{array}$$

It is easy to see that, if $a \in \{1, 3\}$, $n \geq 0$, if

$$C(n,a) \vdash (3n+27+10a) \quad C(2, 3n+3a-1) = C(2, 4q+r) \vdash (t_q) \quad C(u_q, r),$$

then $q \geq (3n-1)/4$ and $u_q > (3^{3/4})^n > 2^n$.

And we also have $n \geq 5 \Rightarrow t_n > 68 \times 9^n$, so, if $C(9,1) \vdash (T)$ END, we have

$$T > t_{q_6} > 9^{q_6} > 9^{3u_{q_5}/4} > 5^{u_{q_5}},$$

and $u_{q_5} > 2^{u_{q_4}}$, $u_{q_4} > 2^{u_{q_3}}$, $u_{q_3} > 2^{u_{q_2}} = 2^{u_{7379}}$, so $T > 5^{2^{2^{2^{u_{7379}}}}}$.

Using $u_{7379} > 10^{3521}$, and, for $x \geq 1$, $2^{10^x} > 10^{10^{x-.53}}$, $2^{10^{10^x}} > 10^{10^{10^{x-.03}}}$, $2^{10^{10^{10^x}}} > 10^{10^{10^{10^{x-.03}}}}$ and $5^{10^{10^{10^{10^x}}}} > 10^{10^{10^{10^{10^{x-.03}}}}}$, we are done. □

8 An infinite set of rules

Let M_6 be the 6×2 Turing machine defined by Table 11.

We have $s(M_6) > 8.9 \times 10^{1762}$ and $\sigma(M_6) > 2.5 \times 10^{881}$.

This machine was discovered in November 2007 by Terry and Shawn Ligocki. It was the champion for the busy beaver competition for 6×2 machines from November to December 2007.

The following theorem gives the rules that enable Turing machine M_6 to reach a halting configuration from a blank tape.

Recall that bin(p) is the usual binary writing of number p, and $\mathrm{R}(w_1 \ldots w_n) = w_n \ldots w_1$.

Theorem 8.1 *Let $C(n,p) = {}^{\omega}0(F0)(10)^n R(bin(p))0^{\omega}$, so that $C(k, 4m+1) = C(k+1, m)$. Then*

(a) $\quad {}^{\omega}0(A0)0^{\omega} \quad \vdash (6) \quad C(0, 15)$, *and, for all* $k, m \geq 0$*,*

(b)	$C(k, 4m+3)$	$\vdash (4k+6)$	$C(k+2, m)$,
(c)	$C(2k, 4m)$	$\vdash (30k^2+20k+15)$	$C(5k+2, 2m+1)$,
(d)	$C(2k+1, 4m)$	$\vdash (30k^2+40k+25)$	$C(5k+2, 32m+20)$,
(e)	$C(k, 8m+2)$	$\vdash (8k+20)$	$C(k+3, 2m+1)$,
(f)	$C(2k, 16m+6)$	$\vdash (30k^2+40k+23)$	$C(5k+2, 32m+20)$,
(g)	$C(2k+1, 16m+6)$	$\vdash (30k^2+80k+63)$	$C(5k+7, 2m+1)$,
(h)	$C(k, 32m+14)$	$\vdash (4k+18)$	$C(k+3, 2m+1)$,
(i)	$C(2k, 128m+94)$	$\vdash (30k^2+40k+39)$	$C(5k+2, 256m+84)$,
(j)	$C(2k+1, 128m+94)$	$\vdash (30k^2+80k+79)$	$C(5k+9, m)$,
(k)	$C(k, 256m+190)$	$\vdash (4k+34)$	$C(k+5, m)$,
(l)	$C(k, 512m+30)$	$\vdash (2k+43)$	${}^{\omega}0(10)^k1(H0)(10)^2(01)^2R(\mathit{bin}(m))0^{\omega}$.

Proof. A direct inspection of the transition table gives

(1)	$0^4(A0)0$	$\vdash (6)$	$(F0)1^40$,
(2)	$0(F0)00$	$\vdash (9)$	$(F0)1^3$,
(3)	$(F0)10$	$\vdash (2)$	$10(F0)$,
(4)	$(F0)11$	$\vdash (4)$	$(F1)10$,
(5)	$(F0)01$	$\vdash (4)$	$10(C1)$,
(6)	$10(E1)$	$\vdash (2)$	$(E1)10$,
(7)	$1(E1)$	$\vdash (2)$	$1(H0)$,
(8)	$0^3(E1)$	$\vdash (3)$	$(F0)110$,
(9)	$00(F1)$	$\vdash (2)$	$(F0)10$,
(10)	$10(F1)$	$\vdash (2)$	$(F1)10$,
(11)	$1(F1)$	$\vdash (1)$	$(E1)0$,
(12)	$(C1)1$	$\vdash (1)$	$0(C1)$,
(13)	$(C1)0$	$\vdash (1)$	$0(C0)$,
(14)	$0^3(C0)$	$\vdash (3)$	$(F0)1^3$,
(15)	$100(C0)$	$\vdash (3)$	$(F1)1^3$.

Iterating, respectively, (3), (10), (6) and (12) gives

(16)	$(F0)(10)^k$	$\vdash (2k)$	$(10)^k(F0)$,
(17)	$(10)^k(F1)$	$\vdash (2k)$	$(F1)(10)^k$,
(18)	$(10)^k(E1)$	$\vdash (2k)$	$(E1)(10)^k$,
(19)	$(C1)1^k$	$\vdash (k)$	$0^k(C1)$.

Using (5), (19) and (13), we get

(20) $(F0)01^{k+1}0 \ \vdash (k+5) \quad 10^{k+2}(C0)$.

Using (16), (4), (17) and (9), we get

(21) $0^2(F0)(10)^k11 \ \vdash (4k+6) \quad (F0)(10)^{k+2}$.

Using (16), (20), (15), (17), (9) and (21), we get

(22) $0^4(F0)(10)^k010 \ \vdash (8k+20) \quad (F0)(10)^{k+3}1$.

Using (22) and (21), we get

(23) $0^6(F0)(10)^k(01)^2 \ \vdash (12k+38) \quad (F0)(10)^{k+5}$.

By induction on n, using (23), we get

(24) $0^{6n}(F0)(10)^k(01)^{2n} \ \vdash (30n^2+12kn+8n) \quad (F0)(10)^{5n+k}$.

Using (24), with $k = 2$, we get

(25) $0^{6k}(F0)(10)^2(01)^{2k} \quad \vdash (30k^2 + 32k) \quad (F0)(10)^{5k+2}$.

Using (16), (2), (4), (11), (18), (8) and (21), we get

(26) $0^5(F0)(10)^{k+1}0^2 \quad \vdash (4k + 25) \quad (F0)(10)^2(01)^k00101$.

Using (26) and (25), we get

(27) $0^{6k+5}(F0)(10)^{2k+1}0^2 \quad \vdash (30k^2 + 40k + 25) \quad (F0)(10)^{5k+2}00101$.

Using (25) and (22), we get

(28) $0^{6k+4}(F0)(10)^2(01)^{2k+1}0 \quad \vdash (30k^2 + 72k + 36) \quad (F0)(10)^{5k+5}1$.

Using (26) and (28), we get

$0^{6k+9}(F0)(10)^{2k+2}0^2 \quad \vdash (30k^2 + 80k + 65) \quad (F0)(10)^{5k+7}1$,

and the result is still true for $k = -1$, so we have

(29) $0^{6k+3}(F0)(10)^{2k}0^2 \quad \vdash (30k^2 + 20k + 15) \quad (F0)(10)^{5k+2}1$.

Using (16), (20), (14), (4), (11), (18), (8) and (21), we get

(30) $0^5(F0)(10)^k01^20 \quad \vdash (4k + 23) \quad (F0)(10)^2(01)^k00101$.

Using (30) and (25), we get

(31) $0^{6k+5}(F0)(10)^{2k}01^20 \quad \vdash (30k^2 + 40k + 23) \quad (F0)(10)^{5k+2}00101$.

Using (30) and (28), we get

(32) $0^{6k+9}(F0)(10)^{2k+1}01^20 \quad \vdash (30k^2 + 80k + 63) \quad (F0)(10)^{5k+7}1$.

Using (16), (20), (14), (4), (17) and (9), we get

(33) $0^2(F0)(10)^k01^30 \quad \vdash (4k + 18) \quad (F0)(10)^{k+3}1$.

Using (16), (20), (14), (4), (9) and (16), we get

(34) $(F0)(10)^k01^40 \quad \vdash (2k + 21) \quad (10)^k1(10)^2(F0)1$.

Using (34), (4), (17), (11), (18), (8) and (21), we get

(35) $0^5(F0)(10)^k01^401 \quad \vdash (4k + 39) \quad (F0)(10)^2(01)^k00(10)^3$.

Using (35) and (25), we get

(36) $0^{6k+5}(F0)(10)^{2k}01^401 \quad \vdash (30k^2 + 40k + 39) \quad (F0)(10)^{5k+2}0^2(10)^3$.

Using (35), (25) and (22), we get

(37) $0^{6k+9}(F0)(10)^{2k+1}01^401 \quad \vdash (30k^2 + 80k + 79) \quad (F0)(10)^{5k+9}$.

Using (16), (20), (14), (4), (9) and (16), we get

(38) $(F0)(01)^k01^50 \quad \vdash (2k + 22) \quad (10)^{k+3}(F0)1$.

Using (38), (4), (17) and (9), we get

(39) $0^2(F0)(10)^k01^501 \quad \vdash (4k + 34) \quad (F0)(10)^{k+5}$.

Using (34), (3), (2), (4), (11), (18) and (7), we get

(40) $(F0)(10)^k01^40^4 \quad \vdash (2k + 43) \quad (10)^k1(H0)(10)^2(01)^2$.

Results (1), (21), (29), (27), (22), (31), (32), (33), (36), (37), (39) and (40) give results (a)–(l) of the theorem. □

Note that the rules (a)–(l) are written in their order of occurrence in the computation of Turing machine M_6 on the blank tape.

Using the rules of this theorem, we have, in 3346 transitions,

$$^\omega0(A0)0^\omega \vdash (6) \quad C(0, 15) \vdash (6) \quad C(2, 3) \vdash (\ ) \cdots \vdash (\ ) \quad \text{END}.$$

M_7	0	1
A	1RB	0LB
B	0RC	1LB
C	1RD	0LA
D	1LE	1LF
E	1LA	0LD
F	1RH	1LE

Table 12: Machine M_7 discovered in October 2000 by Marxen and Buntrock

We have

$$\begin{aligned} & {}^{\omega}0(A0)0\mathrm{R}(\mathrm{bin}(p))0^{\omega} \vdash (6) \quad {}^{\omega}0(F0)1^4 0\mathrm{R}(\mathrm{bin}(p))0^{\omega} \\ & = C(0, 32p+15) \vdash (6) \quad C(2, 8p+3) \vdash (14) \quad C(4, 2p), \end{aligned}$$

so the behavior of Turing machine M_6 on inputs $00x$, $x \in \{0,1\}^*$, depends on the behavior of configurations $C(n,p)$, and the halting problem for Turing machine M_6 depends on this behavior.

9 Configurations provably stopping

Let M_7 be the 6×2 Turing machine defined by Table 12.

We have $s(M_7) > 6.1 \times 10^{925}$ and $\sigma(M_7) > 6.4 \times 10^{462}$.

This machine was discovered in October 2000 by Heiner Marxen and Jürgen Buntrock. It was the champion for the busy beaver competition for 6×2 machines from October 2000 to March 2001.

The following theorem was initially obtained by Munafo [37]. It gives the rules observed by Turing machine M_7.

Theorem 9.1 *Let $C(n) = {}^{\omega}01^n(B0)0^{\omega}$. Then*

(a) ${}^{\omega}0(A0)0^{\omega} \ \vdash (1) \quad C(1),$

and, for all $k \geq 0$,

(b) $C(3k) \quad \vdash (54 \times 4^{k+1} - 27 \times 2^{k+3} + 26k + 86) \quad C(9 \times 2^{k+1} - 8),$

(c) $C(3k+1) \quad \vdash (2048 \times (4^k - 1)/3 - 3 \times 2^{k+7} + 26k + 792) \quad C(2^{k+5} - 8),$

(d) $C(3k+2) \quad \vdash (3k+8) \quad {}^{\omega}01(H1)(011)^k 01010^{\omega}.$

Proof. A direct inspection of the transition table gives

(1) $11(B0)00 \ \vdash (6) \ (D1)0101,$
(2) $00(B0)01 \ \vdash (8) \ (B0)1^4,$
(3) $11(B0)01 \ \vdash (6) \ (B1)01^3,$
(4) $(B0)1 \ \vdash (3) \ 1(B0),$
(5) $0(B1) \ \vdash (1) \ (B0)1,$
(6) $1(B1) \ \vdash (1) \ (B1)1,$
(7) $0(D1) \ \vdash (2) \ 1(H1),$
(8) $0^3 1(D1) \ \vdash (6) \ (B0)1^4,$
(9) $0^4 1^2(D1) \ \vdash (8) \ (B0)1^3 01^2,$
(10) $1^3(D1) \ \vdash (3) \ (D1)011.$

Iterating, respectively, (4), (6) and (10) gives

(11) $(B0)1^k \quad \vdash (3k) \quad 1^k(B0)$,

(12) $1^k(B1) \quad \vdash (k) \quad (B1)1^k$,

(13) $1^{3k}(D1) \quad \vdash (3k) \quad (D1)(011)^k$.

Using (3), (12), (5) and (11), we get

$01^{k+2}(B0)01 \quad \vdash (4k+10) \quad 1^{k+1}(B0)01^3$,

and the result is still true for $k=-1$, so we have

(14) $01^{k+1}(B0)01 \quad \vdash (4k+6) \quad 1^k(B0)01^3$.

By induction on k, using (14), we get

(15) $0^k1^k(B0)01 \quad \vdash (2k^2+4k) \quad (B0)01^{2k+1}$.

Using (1), (13) and (7), we get

(16) $01^{3k+2}(B0)0^2 \quad \vdash (3k+8) \quad 1(H1)(011)^k0101$.

Using (1), (13) and (9), we get

$0^41^{3k+4}(B0)0^2 \quad \vdash (3k+14) \quad (B0)1^3(011)^{k+1}0101$,

and the result is still true for $k=-1$, so we have

(17) $0^41^{3k+1}(B0)0^2 \quad \vdash (3k+11) \quad (B0)1^3(011)^k0101$.

Using (1), (13) and (8), we get

(18) $0^31^{3k+3}(B0)0^2 \quad \vdash (3k+12) \quad (B0)1^4(011)^k0101$.

Using (11), (15) and (2), we get

(19) $0^{k+2}(B0)1^k01 \quad \vdash (2k^2+7k+8) \quad (B0)1^{2k+4}$.

By induction on k, using (19), we get

(20) $0^{2^k(n+5)-5-n-3k}(B0)1^n(011)^k \quad \vdash (T) \quad (B0)1^{2^k(n+5)-5}$,

with $T = 2(n+5)^2(4^k-1)/3 - 13(n+5)(2^k-1) + 23k$.

Using (20), for $n=3$ and $n=4$, we get respectively

(21) $0^{2^{k+3}-3k-8}(B0)1^3(011)^k \quad \vdash (128(4^k-1)/3 - 13\times 2^{k+3} + 23k + 104) \quad (B0)1^{2^{k+3}-5}$,

(22) $0^{9\times 2^k-3k-9}(B0)1^4(011)^k \quad \vdash (54\times 4^k - 117\times 2^k + 23k + 63) \quad (B0)1^{9\times 2^k-5}$.

Using (11), (15), (2), (11), (15), (2) and (11), we get

(23) $0^{3k+8}(B0)1^k0101 \quad \vdash (10k^2+65k+112) \quad 1^{4k+12}(B0)$.

Using (17), (21) and (23), we get

(24) $0^{2^{k+5}-3k-11}1^{3k+1}(B0)0^2 \quad \vdash (2048\times(4^k-1)/3 - 3\times 2^{k+7} + 26k + 792) \quad 1^{2^{k+5}-8}(B0)$.

Using (18), (22) and (23), we get

$0^{9\times 2^{k+2}-3k-13}1^{3k+3}(B0)0^2 \quad \vdash (54\times 4^{k+2} - 27\times 2^{k+4} + 26k + 112) \quad 1^{9\times 2^{k+2}-8}(B0)$,

and the result is still true for $k=-1$, so we have

(25) $0^{9\times 2^{k+1}-3k-10}1^{3k}(B0)0^2 \quad \vdash (54\times 4^{k+1} - 27\times 2^{k+3} + 26k + 86) \quad 1^{9\times 2^{k+1}-8}(B0)$.

Results (25), (24) and (16) give results (b)–(d) of the theorem. □

Using the rules of this theorem, we have

$$^\omega 0(A0)0^\omega \vdash (1) \quad C(1) \vdash (408) \quad C(24) \vdash (14100774)$$

$$C(4600) \vdash (T) \quad C(2^{1538}-8) \vdash (2^{1538}-2) \quad {}^\omega 01(H1)(011)^p01010^\omega,$$

with $T = 2048\times(4^{1533}-1)/3 - 3\times 2^{1540} + 40650$ and $p = (2^{1538}-10)/3$.

So the total time is $s(M_7) = 2048 \times (4^{1533} - 1)/3 - 11 \times 2^{1538} + 14141831$, and the final number of symbols 1 is $\sigma(M_7) = 2 \times (2^{1538} - 10)/3 + 4$.

Note that

$$C(6k+1) \vdash (\ ) \quad C(3m) \vdash (\ ) \quad C(6p+4) \vdash (\ ) \quad C(3q+2) \vdash (\ ) \quad \text{END},$$

with $m = (2^{2k+5} - 8)/3$, $p = 3 \times 2^m - 2$, $q = (2^{2p+6} - 10)/3$. So all configurations $C(n)$ lead to a halting configuration. Those taking the most time are $C(6k+1)$. For example, $C(7) \vdash (t)$ END with $t > 10^{3.9 \times 10^{12}}$. More generally, $C(6k+1) \vdash (t(k))$ END with $t(k) > 10^{10^{10^{(3k+2)/5}}}$.

10 Conclusion

We discuss two questions as a conclusion to this article.

A. *How simulating Collatz-like functions allows Turing machines to achieve high scores?*

Lagarias [17] noted that the successive iterates of the $3x+1$ function T have an irregular behavior. For example, 7 iterations of function T on $n = 26$ lead to the value 1, but 70 iterations are necessary on $n = 27$. It seems that many Collatz-like functions have the same irregular behavior. Iterating them on small numbers may produce very long runs before stopping.

Adding parameters may increase the number of iterations by allowing the iterated values to range the set of parameters before stopping. The pure Collatz-like function with parameter $g_3(n, p)$ presented in Section 5 is particularly illustrative.

Another way to high scores is given by exponential Collatz-like functions such as function g_4 in Section 6. Only five iterations are performed on a blank tape, but exponential growth ensures a high score.

Irregular behavior is a condition for a Collatz-like function to be eligible to the busy beaver competition. Another condition is, of course, being computable by a very small Turing machine.

B. *Are some universal devices more natural than others?*

Conway [7] proved that there is no algorithm that, given as inputs a Collatz-like function g and two integers n, p, outputs an answer yes or no to the question: Does there exist a positive integer k such that $g^k(n) = p$? Conway [7, 8] also proved that Collatz-like functions can be used to simulate all computable (also called recursive) functions. These properties can be summed up by writing that Collatz-like functions provide a universal model of computation with a m-complete decision problem.

Many universal models of computation are known: Turing machines, tag-systems, cellular automata, Diophantine equations, etc. (see [35]). Of course, any universal model can simulate and be simulated by any other universal model. But it is Collatz-like functions, and not another model, that appear naturally in this study. Their unexpectedly pervasive presence leads to wonder about the significance of their status among mathematical beings.

References

[1] N. Bátfai, On the running time of the shortest programs, http://arxiv.org/abs/0908.1159.

[2] N. Bátfai, Recombinations of busy beaver machines, http://arxiv.org/abs/0908.4013.

[3] A.H. Brady, The conjectured highest scoring machines for Rado's $\Sigma(k)$ for the value $k = 4$, *IEEE Transactions on Electronic Computers*, EC-**15**, October 1966, 802–803.

[4] A.H. Brady, The determination of the value of Rado's noncomputable function $\Sigma(k)$ for four-state Turing machines, *Mathematics of Computation*, **40** (162), April 1983, 647–665.

[5] A.H. Brady, The busy beaver game and the meaning of life, in: The Universal Turing machine: A Half-Century Survey, R. Herken (Ed.), Oxford University Press, 1988, 259–277.

[6] R.N. Buttsworth and K.R. Matthews, On some Markov matrices arising from the generalized Collatz mapping, *Acta Arithmetica* **55**, 1990, 43–57.

[7] J.H. Conway, Unpredictable iterations, in: *Proc. 1972 Number Theory Conference*, University of Colorado, Boulder, CO, 1972, 49–52, and also in [20], 219–221.

[8] J.H. Conway, FRACTAN: a simple universal programming language for arithmetic, in: *Open problems in Communication and Computation* (T.M. Cover and B. Gopinath, Eds.), Springer, 1987, 4–26, and also in [20], 249–264.

[9] A.K. Dewdney, Computer recreations, *Scientific American* **251** (5), November 1984, 27.

[10] G. Georgiev (Skelet), http://skelet.ludost.net/bb.

[11] M.W. Green, A lower bound on Rado's sigma function for binary Turing machines, in: *Proceedings of the 5th IEEE Annual Symposium on Switching Circuit Theory and Logical Design*, November 1964, 91–94.

[12] J. Hertel, Computing the uncomputable Rado sigma function, *The Mathematica Journal* **11** (2), 2009, 270–283.

[13] T. Hutton, Two dimensional Turing machines, http://code.google.com/p/ruletablerepository/wiki/TwoDimensionalTuringMachines.

[14] F. Kaščák, Small universal one-state linear operator algorithm, Proc. MFCS'92, *Lecture Notes in Computer Science* **629**, 1992, 327–335.

[15] S. Kohl, Wildness of iteration of certain residue-class-wise affine mappings, *Advances in Applied Mathematics* **39** (3), 2007, 322–328.

[16] G. Lafitte and C. Papazian, The fabric of small Turing machines, in: *Computation and Logic in the Real World, Proceedings of the Third Conference on Computability in Europe*, June 2007, 219–227.

[17] J.C. Lagarias, The $3x+1$ problem and its generalizations, *Amer. Math. Monthly* **92** (1), 1985, 3–23, and also in [20], 31–54.

[18] J.C. Lagarias, The $3x+1$ problem: an annotated bibliography (1963–1999), in [20], 267–341, and also http://arxiv.org/abs/math/0309224.

[19] J.C. Lagarias, The $3x+1$ problem: an annotated bibliography (2000–), http://arxiv.org/abs/math/0608208.

[20] J.C. Lagarias (Ed.), The Ultimate Challenge: The $3x+1$ Problem, AMS, 2010.

[21] T. Ligocki, Communicated by email on 13 February 2005.

[22] S. Ligocki, Communicated by email on 15 August 2006.

[23] S. Ligocki, Communicated by email on 20 August 2006.

[24] S. Ligocki, Communicated by email on 8 May 2010.

[25] S. Lin and T. Rado, Computer studies of Turing machine problems, *Journal of the ACM* **12** (2), April 1965, 196–212.

[26] D.S. Lynn, New results for Rado's sigma function for binary Turing machines, *IEEE Transactions on Computers*, C-**21** (8), August 1972, 894–896.

[27] R. Machlin and Q.F. Stout, The complex behavior of simple machines, *Physica D* **42**, June 1990, 85–98.

[28] H. Marxen, http://www.drb.insel.de/~heiner/BB/macro.html.

[29] H. Marxen and J. Buntrock, Attacking the Busy Beaver 5, *Bulletin of the EATCS*, No 40, February 1990, 247–251.

[30] P. Michel, Étude de machines de Turing et complexité algorithmique, Thèse Université Paris 7, 1992.

[31] P. Michel, Busy beaver competition and Collatz-like problems, *Arch. Math. Logic* **32** (5), 1993, 351–367.

[32] P. Michel, The busy beaver competition: a historical survey, http://arxiv.org/abs/0906.3749.

[33] P. Michel, Historical survey of busy beavers, http://www.logique.jussieu.fr/~michel/ha.html.

[34] P. Michel, Behavior of busy beavers, http://www.logique.jussieu.fr/~michel/beh.html.

[35] P. Michel and M. Margenstern, Generalized $3x+1$ functions and the theory of computation, in [20], 105–128.

[36] R. Munafo, http://mrob.com/pub/math/ln-notes1-4.html#mb-bb-1.

[37] R. Munafo, http://mrob.com/pub/math/ln-notes1-5.html#mb6q, and http://mrob.com/pub/math/ln-mb6q.html.

[38] T. Rado, On non-computable functions, *Bell Syst. Tech. J.* **41** (3), May 1962, 877–884.

[39] D.A. Rawsthorne, Imitation of an iteration, *Mathematics Magazine* **58** (3), May 1985, 172–176.

[40] C. Tooth, Google Groups sci.math>the Turing machine known as #r, June 28, 2002.

[41] S. Wagon, The Collatz problem, *The Mathematical Intelligencer* **7** (1), 1985, 72–76.

[42] G.J. Wirsching, The Dynamical System Generated by the 3n+1 Function, *Lecture Notes in Mathematics* **1681**, Springer, 1998.